This report is a supplementary document to the paper "Filter-and-forward relay design for MIMO-OFDM systems," by Donggun Kim, Youngchul Sung, and Jihoon Chung, accepted to IEEE Transactions on Communications.

Proof of Lemma 1

Note that Lemma 1 is modified from [R.1]. To prove Lemma 1, we start with verifying the fact that a block circulant matrix is a polynomial in the permutation matrix $\mathbf{E}$, where a polynomial in $\mathbf{E}$ is defined as

$$\mathbf{H}_c = (\mathbf{I}_N \otimes \mathbf{H}_0) + (\mathbf{E} \otimes \mathbf{H}_1) + (\mathbf{E}^2 \otimes \mathbf{H}_2) + \cdots + (\mathbf{E}^{N-1} \otimes \mathbf{H}_{N-1}) \tag{1}$$

where $\mathbf{E} = [\mathbf{e}_N \ \mathbf{e}_1 \ \mathbf{e}_2 \ \cdots \ \mathbf{e}_{N-1}]$ and $\mathbf{e}_i$ is the i-th column of the $N \times N$ identity matrix $\mathbf{I}_N$. Using this, we can rewrite $\mathbf{\Lambda}_b$ as

$$\begin{aligned}
\mathbf{\Lambda}_b &= (\mathbf{W}_N^H \otimes \mathbf{I}_{N_r})\, \mathbf{H}_c\, (\mathbf{W}_N \otimes \mathbf{I}_{N_t}) && (2)\\
&= (\mathbf{W}_N^H \otimes \mathbf{I}_{N_r})\, \big((\mathbf{I}_N \otimes \mathbf{H}_0) + (\mathbf{E} \otimes \mathbf{H}_1) + \cdots + (\mathbf{E}^{N-1} \otimes \mathbf{H}_{N-1})\big)\, (\mathbf{W}_N \otimes \mathbf{I}_{N_t}) && (3)\\
&= (\mathbf{W}_N^H \otimes \mathbf{I}_{N_r})(\mathbf{I}_N \otimes \mathbf{H}_0)(\mathbf{W}_N \otimes \mathbf{I}_{N_t}) + (\mathbf{W}_N^H \otimes \mathbf{I}_{N_r})(\mathbf{E} \otimes \mathbf{H}_1)(\mathbf{W}_N \otimes \mathbf{I}_{N_t}) + \cdots \\
&\qquad + (\mathbf{W}_N^H \otimes \mathbf{I}_{N_r})(\mathbf{E}^{N-1} \otimes \mathbf{H}_{N-1})(\mathbf{W}_N \otimes \mathbf{I}_{N_t}) && (4)\\
&\overset{(a)}{=} (\mathbf{W}_N^H \otimes \mathbf{H}_0)(\mathbf{W}_N \otimes \mathbf{I}_{N_t}) + \cdots + (\mathbf{W}_N^H \mathbf{E}^{N-1} \otimes \mathbf{H}_1)(\mathbf{W}_N \otimes \mathbf{I}_{N_t}) && (5)\\
&\overset{(b)}{=} (\mathbf{W}_N^H \mathbf{W}_N \otimes \mathbf{H}_0) + \cdots + (\mathbf{W}_N^H \mathbf{E}^{N-1} \mathbf{W}_N \otimes \mathbf{H}_{N-1}) && (6)
\end{aligned}$$

where (a) and (b) follow from the Kronecker product identity that $(\mathbf{A} \otimes \mathbf{C})(\mathbf{B} \otimes \mathbf{D}) = (\mathbf{A}\mathbf{B}) \otimes (\mathbf{C}\mathbf{D})$. Note that $\mathbf{E}^k$ is a circulant matrix for $k \in \mathbb{N}$ and this can be diagonalized by $\mathbf{W}_N^H$ and $\mathbf{W}_N$. Therefore, $\mathbf{\Lambda}_b$ is a block diagonal matrix. Let us first consider $\mathbf{W}_N^H \mathbf{E} \mathbf{W}_N$. The first row of $\mathbf{E}$ is $\mathbf{e}_2^T = [0 \ \ 1 \ \ 0 \ \ 0 \ \ \cdots \ \ 0]$. Then, $\mathbf{E}$ can be diagonalized by the circulant matrix theorem and its diagonal elements is given as

$$\sqrt{N}\mathbf{W}_N^H \mathbf{e}_2 = \left[\, 1 \ \omega_N^{-1} \ \omega_N^{-2} \ \cdots \ \omega_N^{-(N-1)} \right]^T \tag{7}$$

where $\omega_N = e^{\iota \frac{2\pi}{N}}$. Therefore, we obtain

$$\mathbf{W}_N^H \mathbf{E} \mathbf{W}_N = \mathbf{D}_E = \text{diag}\{\sqrt{N}\mathbf{W}_N^H \mathbf{e}_2\} = \text{diag}\left\{ \left[\, 1 \ \omega_N^{-1} \ \omega_N^{-2} \ \cdots \ \omega_N^{-(N-1)} \right]^T \right\} \tag{8}$$

Next, let us consider $\mathbf{W}_N^H\mathbf{E}^2\mathbf{W}_N$. The first row of $\mathbf{E}^2$ is $\mathbf{e}_3^T = [0 \;\; 0 \;\; 1 \;\; 0 \;\; \cdots \;\; 0]$ because $\mathbf{E}^2$ is a column-shifted version of $\mathbf{E}$. $\mathbf{E}^2$ can also be diagonalized by the circulant matrix theorem and its diagonal elements is given as

$$\sqrt{N}\mathbf{W}_N^H\mathbf{e}_3 = \left[\, 1 \; \omega_N^{-2} \; \omega_N^{-4} \; \cdots \; \omega_N^{-2(N-1)} \right]^T. \tag{9}$$

Therefore, we obtain $\mathbf{W}_N^H\mathbf{E}^2\mathbf{W}_N = \mathbf{D}_E^2$. In a similar way, we can generalize the diagonalization of permutation matrices as

$$\mathbf{W}_N^H\mathbf{E}^n\mathbf{W}_N = \mathbf{D}_E^n. \tag{10}$$

From this, the equation (6) can be rewritten as

$$\begin{aligned}
\boldsymbol{\Lambda}_b &= (\mathbf{W}_N^H\mathbf{W}_N \otimes \mathbf{H}_0) + \cdots + (\mathbf{W}_N^H\mathbf{E}^{N-1}\mathbf{W}_N \otimes \mathbf{H}_{N-1}) \\
&= \sum_{n=0}^{N-1} \mathbf{W}_N^H\mathbf{E}^n\mathbf{W}_N \otimes \mathbf{H}_n \qquad (11)\\
&= \sum_{n=0}^{N-1} \mathbf{D}_E^n \otimes \mathbf{H}_n \qquad (12)
\end{aligned}$$

and the i-th diagonal block of $\boldsymbol{\Lambda}_b$ is given by

$$\sum_{n=0}^{N-1} (\mathbf{D}_E^n)_{ii}\, \mathbf{H}_n = \sum_{n=0}^{N-1} \mathbf{H}_n\, \omega_N^{-n(i-1)} = \sum_{n=0}^{N-1} \mathbf{H}_n\, e^{-\iota 2\pi \frac{n(i-1)}{N}}.$$

Let us define the i-th row of DFT matrix $\sqrt{N}\mathbf{W}_N^H$ as $\sqrt{N}\mathbf{w}_{N-i}^H = [1 \; \omega_N^{-(i-1)} \; \omega_N^{-2(i-1)} \; \cdots \; \omega_N^{-(N-1)(i-1)}]$ and $\mathbf{K} = [\mathbf{H}_0 \; \mathbf{H}_1 \; \cdots \; \mathbf{H}_{N-1}]$. Then, we can rewrite the i-th diagonal block as $\mathbf{K}(\sqrt{N}\mathbf{w}_{N-i}^H \otimes \mathbf{I}_{N_t})^T$. If we replace i with $-(k-N)$, we obtain the desired result. □

[R.1] P. J. Davis, *Circulant Matrices*, 2nd ed. New York: Chelsea, 1994.