Outage Probability and Outage-Based Robust Beamforming for MIMO Interference Channels with Imperfect Channel State Information

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***Abstract*—In this paper, the outage probability and outage-based beam design for multiple-input multiple-output (MIMO) interference channels are considered. First, closed-form expressions for the outage probability in MIMO interference channels are derived under the assumption of Gaussian-distributed channel state information (CSI) error, and the asymptotic behavior of the outage probability as a function of several system parameters is examined by using the Chernoff bound. It is shown that the outage probability decreases exponentially with respect to the quality of CSI measured by the inverse of the mean square error of CSI. Second, based on the derived outage probability expressions, an iterative beam design algorithm for maximizing the sum outage rate is proposed. Numerical results show that the proposed beam design algorithm yields significantly better sum outage rate performance than conventional algorithms such as interference alignment developed under the assumption of perfect CSI.**

***Index Terms*—Multiuser MIMO, interference channels, channel uncertainty, outage probability, Chernoff bound, interference alignment.**

I. Introduction

Due to their importance in current and future wireless communication systems, multiple-input multiple-output (MIMO) interference channels have gained much attention from the research community in recent years. Since Cadambe and Jafar showed that interference alignment (IA) achieved the maximum number of degrees of freedom in MIMO interference channels [2], there has been extensive research in devising good beam design algorithms for MIMO interference channels. Now, there are many available beam design algorithms for MIMO interference channels such as IA-based algorithms [3]–[5] and sum-rate targeted algorithms [3], [4], [6]–[9]. However, most of these algorithms assume perfect channel state information (CSI) at transmitters and receivers, whereas the assumption of perfect CSI is unrealistic in practical wireless communication systems since perfect CSI is typically unavailable in practical systems due to channel estimation error, limited feedback or other limitations [10]. Thus, the CSI error should be incorporated into the beam design to yield better performance, and this is typically done under robust beam design frameworks.

There are many robust beam design studies in the conventional single-user MIMO case and also in the multiple-input and single-output (MISO) multi-user case. In the MISO multi-user case, the problem is more tractable than in the MIMO multi-user case, and extensive research results are available on MISO broadcast and interference channels with imperfect CSI [11]–[13]; the outage rate region is defined for MISO interference channels in [11], and the optimal beam structure that achieves a Pareto-optimal point of the outage rate region is given in [12]. For more complicated MIMO interference channels, there are several pioneering works on robust beam design under CSI uncertainty [14]–[16]. In [14], the authors solved the problem based on a worst-case approach. In their work, the CSI error is modelled as a random variable under a Frobenius norm constraint, and a semi-definite relaxation method is used to obtain the beam vectors that maximize the minimum signal-to-interference-plus-noise ratio (SINR) over all users and all possible CSI error. In [15], on the other hand, the CSI error is modelled as an independent Gaussian random variable, and the beam is designed to minimize the mean square error (MSE) between the transmitted signal and the reconstructed signal at the receiver with given imperfect CSI at the transmitter (CSIT).

In this paper, we consider robust beam design in MIMO interference channels based on a different criterion. Here, we consider the rate outage due to channel uncertainty and the problem of sum rate maximization under an outage constraint in MIMO interference channels. This formulation is practically meaningful since an outage probability is assigned to each user and the supportable rate with the given outage probability is maximized. Here, we assume that the transmitters and receivers have imperfect CSI and the CSI error is circularly-symmetric complex Gaussian distributed. Under this assumption, we first derive closed-form expressions for the outage probability in MIMO interference channels for an arbitrarily given set of transmit and receive beamforming vectors, and then derive the asymptotic behavior of the outage probability as a function of several system parameters by using the Chernoff bound. It is shown that *the outage probability*

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decreases exponentially with respect to (w.r.t.) the quality of CSI measured by the inverse of the mean square error (MSE) of CSI, typically called the channel K factor [10] or interpreted as the Fisher information [17] in statistical estimation theory. In particular, it is shown that in the case of interference alignment, the outage probability can be made arbitrarily small by improving the CSI quality if the target rate is strictly less than the rate obtained by using the estimated channel as the nominal channel. Next, based on the derived outage probability expressions, we propose an iterative beam design algorithm for maximizing the weighted sum rate under the constraint that the outage probability for each user is less than a certain level. Numerical results show that the proposed beam design algorithm yields better sum outage rate performance than conventional beam design algorithms such as the 'max-SINR' algorithm [3] developed without the consideration of channel uncertainty.

A. Related work

The outage analysis for MIMO interference channels has been performed by several other researchers [16], [18]. In [16], the outage probability for a given rate tuple is computed under the assumption that knowledge of the channel mean and covariance matrix is available, and transmit and receive beam vectors that minimize the power consumption for a given outage constraint are obtained. However, it is difficult to generalize this method of analysis to the case of multiple data streams per user, whereas our analysis includes the multiple data stream case. In [18], the outage probability and SINR distribution of each user in MIMO interference channels with the knowledge of channel distribution information are obtained under a particular transmit and receive beam structure of IA transmit beams and zero-forcing (ZF) receivers. On the other hand, our analysis can be applied to the case of general transmit and receive beam structures beyond IA and ZF.

The probability distribution of a quadratic form of Gaussian random variables has been studied extensively in statistics [19]–[22] and in communications [23]–[25]. The most widely-used approach to obtain the probability distribution of a Gaussian quadratic form is the series fitting method [20], [21], [23], [26], which typically converges to the probability distribution of a Gaussian quadratic form from the lower tail first. However, the outage definition associated with robust beam design for MIMO interference channels in this paper requires accurate computation of upper tail probabilities. The series expansion for the cumulative distribution function (CDF) obtained in this paper based on the integral form for the CDF in [25] and the residue theorem [22] is well suited to this purpose and converges to the upper tail first. Thus, the obtained series in this paper is more relevant for our outage analysis. For a detailed explanation of the derived series, please refer to [27].

B. Notation and organization

We will make use of standard notational conventions. Vectors and matrices are written in boldface with matrices in capitals. All vectors are column vectors. For a matrix $\mathbf{A}$, $\mathbf{A}^H$, $\|\mathbf{A}\|_F$ and $\mathbf{A}(i,j)$ indicate the Hermitian transpose, the Frobenius norm and the element in row i and column j of $\mathbf{A}$, respectively, and $\text{vec}(\mathbf{A})$ and $\text{tr}(\mathbf{A})$ denote the vector composed of the columns of $\mathbf{A}$ and the trace of $\mathbf{A}$, respectively. For vector $\mathbf{a}$, $\|\mathbf{a}\|$ and $[\mathbf{a}]_i$ represent the 2-norm and the i-th element of $\mathbf{a}$, respectively. $\mathbf{I}_n$ stands for the identity matrix of size n (the subscript is included only when necessary), and $\text{diag}(d_1, \cdots, d_n)$ means a diagonal matrix with diagonal elements $d_1, \cdots, d_n$. $\mathbf{x} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ means that the random vector $\mathbf{x}$ has the circularly-symmetric complex Gaussian distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. $\mathcal{K} = \{1, 2, \cdots, K\}$, $\iota = \sqrt{-1}$, and $|A|$ denotes the cardinality of the set A.

The paper is organized as follows. The system model and problem formulation are described in Section II. In Section III, closed-form expressions for the outage probability are derived, and the behavior of the outage probability as a function of several system parameters is examined by using the Chernoff bound. In Section IV, an outage-based beam design algorithm is proposed. Numerical results are provided in Section V, followed by the conclusion in Section VI.

II. System Model and Problem Formulation

In this paper, we consider a K-user time-invariant MIMO interference channel in which each transmitter equipped with N_t antennas is paired with a receiver equipped with N_r antennas, and interferes with all receivers other than the desired receiver. We assume that transmitter k transmits d ($\leq \min(N_t, N_r)$) independent data streams to receiver k paired with transmitter k. Then, the received signal at receiver k is given by

$$\mathbf{y}_k = \mathbf{H}_{kk}\mathbf{V}_k\mathbf{s}_k + \sum_{i=1, i\neq k}^{K} \mathbf{H}_{ki}\mathbf{V}_i\mathbf{s}_i + \mathbf{n}_k, \tag{1}$$

where $\mathbf{H}_{ki}$ is the $N_r \times N_t$ channel matrix from transmitter i to receiver k; $\mathbf{V}_i = [\mathbf{v}_i^{(1)}, \cdots, \mathbf{v}_i^{(d)}]$ is the $N_t \times d$ transmit beamforming matrix with normalized column vectors at transmitter i, i.e., $||\mathbf{v}_i^{(m)}|| = 1$ for $m = 1, \cdots, d$; and $\mathbf{s}_i = [s_i^{(1)}, \cdots, s_i^{(d)}]^T$ is the $d \times 1$ symbol vector at transmitter i. We assume that the transmit symbol vector $\mathbf{s}_i$ is drawn from the zero-mean Gaussian distribution with unit variance, i.e., $\mathbf{s}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$, and the additive noise vector $\mathbf{n}_k$ is zero-mean Gaussian distributed with variance σ^2, i.e., $\mathbf{n}_k \sim \mathcal{CN}(\mathbf{0}, \sigma^2\mathbf{I})$. We assume that the CSI available to the system is not perfect. That is, neither the transmitters nor the receivers have perfect CSI. For the imperfect CSI, we adopt the following model

$$\mathbf{H}_{ki} = \hat{\mathbf{H}}_{ki} + \mathbf{E}_{ki} \tag{2}$$

for each $(k, i) \in \mathcal{K} \times \mathcal{K}$, where $\mathbf{H}_{ki}$ is the unknown true channel, $\hat{\mathbf{H}}_{ki}$ is the channel state available to the transmitters and the receivers, and $\mathbf{E}_{ki}$ is the error between the true and available channel information. For the CSI error $\mathbf{E}_{ki}$ between the true and available channel information, we adopt the Kronecker error model which is widely used for MIMO systems to model the error correlation that may be caused by the transmit and receive antenna structure [10]. Under this model, the CSI error $\mathbf{E}_{ki}$ is given by

$$\mathbf{E}_{ki} = \boldsymbol{\Sigma}_r^{1/2}\mathbf{H}_{ki}^{(w)}\boldsymbol{\Sigma}_t^{1/2}, \tag{3}$$

with $\text{vec}(\mathbf{H}_{ki}^{(w)}) \sim \mathcal{CN}(0, \sigma_h^2 \mathbf{I})$ for some $\sigma_h^2 \geq 0$, where $\mathbf{\Sigma}_t$ and $\mathbf{\Sigma}_r$ are transmit and receive antenna correlation matrices, respectively, and the elements of $\mathbf{H}_{ki}^{(w)}$ are independent and identically distributed (i.i.d.) and are drawn from a circularly-symmetric zero-mean complex Gaussian distribution. The CSI uncertainty matrix $\mathbf{E}_{ki}$ is a circularly-symmetric[1] complex Gaussian random matrix with distribution $\text{vec}(\mathbf{E}_{ki}) \sim \mathcal{CN}(\mathbf{0}, \sigma_h^2(\mathbf{\Sigma}_t^T \otimes \mathbf{\Sigma}_r))$ [10, p.90], and σ_h^2 is the parameter capturing the uncertainty level in CSI. We assume that the $\mathbf{E}_{ki}$'s are independent across transmitter-receiver pairs (k, i). To specify the quality of CSI and signal reception, we define two parameters

$$K_{ch}^{(ki)} := \frac{\|\hat{\mathbf{H}}_{ki}\|_F^2}{\mathbb{E}\{\|\mathbf{E}_{ki}\|_F^2\}} = \frac{\|\hat{\mathbf{H}}_{ki}\|_F^2}{\sigma_h^2 \text{tr}(\mathbf{\Sigma}_t^T \otimes \mathbf{\Sigma}_r)} \tag{4}$$

and

$$\Gamma^{(k)} := \frac{\|\hat{\mathbf{H}}_{kk}\|_F^2}{\sigma^2}. \tag{5}$$

$K_{ch}^{(ki)}$ is the channel K factor defined as the ratio of the power of the known channel part to that of the unknown channel part, representing the quality of CSI [10], and $\Gamma^{(k)}$ is the signal-to-noise ratio (SNR) at receiver k since $\mathbf{V}_k$ and $\mathbf{s}_k$ are normalized in our formulation. Hereafter, we will use $\hat{\mathcal{H}}$ to represent the collection of channel information $\{\hat{\mathbf{H}}_{ki}, \mathbf{\Sigma}_t, \mathbf{\Sigma}_r\}$ known to the transmitters and receivers. By using the receiver filter $\mathbf{u}_k^{(m)}$ ($||\mathbf{u}_k^{(m)}|| = 1$), receiver k projects the received signal $\mathbf{y}_k$ in (1) to recover the desired signal stream m:

$$\begin{aligned} \hat{s}_k^{(m)} &= (\mathbf{u}_k^{(m)})^H \mathbf{y}_k \\ &= (\mathbf{u}_k^{(m)})^H \bigg((\hat{\mathbf{H}}_{kk} + \mathbf{E}_{kk}) \mathbf{V}_k \mathbf{s}_k + \sum_{i=1, i \neq k}^{K} (\hat{\mathbf{H}}_{ki} + \mathbf{E}_{ki}) \mathbf{V}_i \mathbf{s}_i + \mathbf{n}_k \bigg). \end{aligned}$$

We assume that the design of the transmit beamforming matrices $\{\mathbf{V}_k, k \in \mathcal{K}\}$ and receive filters $\{\mathbf{U}_k = [\mathbf{u}_k^{(1)}, \cdots, \mathbf{u}_k^{(d)}], k \in \mathcal{K}\}$ is based on the available CSI $\hat{\mathcal{H}}$. This model of beam design and signal transmission and reception captures many coherent linear beamforming MIMO schemes including interference alignment and sum rate maximizing beamforming schemes [3], [6], [28] in which transmit and receive beamforming matrices are designed based on available CSI at transmitters and receivers. Under this processing model, the SINR for stream m of user k is given by (6) on the next page where the numerator of the right-hand side (RHS) of (6) is the desired signal power, and the first, second, third and fourth terms in the denominator of the RHS of (6) represent the interference purely by channel uncertainty, inter-stream interference, other user interference and thermal noise, respectively. (Here, the dependence of SINR on $\hat{\mathcal{H}}$ is explicitly shown. Since the dependence is clear, the notation $|_{\hat{\mathcal{H}}}$ will be omitted hereafter.) Because the $\{\mathbf{E}_{ki}\}$ are random, $\mathsf{SINR}_k^{(m)}$ is a random variable for given $\hat{\mathcal{H}}$ and $\{\mathbf{V}_k(\hat{\mathcal{H}}), \mathbf{U}_k(\hat{\mathcal{H}}), k \in \mathcal{K}\}$. Thus, an outage at stream m of user k occurs if the supportable rate determined by the received SINR (6) is below the target rate $R_k^{(m)}$, and the outage probability is given by

$$\Pr\{\text{outage}\} = \Pr\left\{\log_2\left(1 + \mathsf{SINR}_k^{(m)}\right) \leq R_k^{(m)}\right\}. \tag{7}$$

By rearranging the terms in (6), the outage event can be expressed as

$$\sum_{i=1}^{K} \sum_{j=1}^{d} X_{ki}^{(mj)H} X_{ki}^{(mj)} \geq \frac{|\mathbf{u}_k^{(m)H} \hat{\mathbf{H}}_{kk} \mathbf{v}_k^{(m)}|^2}{2^{R_k^{(m)}} - 1} - \sigma^2 =: \tau, \tag{8}$$

where

$$X_{ki}^{(mj)} := \begin{cases} \mathbf{u}_k^{(m)H} \mathbf{E}_{kk} \mathbf{v}_k^{(m)}, & i = k \ \text{ and } j = m, \\ \mathbf{u}_k^{(m)H} (\hat{\mathbf{H}}_{ki} + \mathbf{E}_{ki}) \mathbf{v}_i^{(j)}, & \text{otherwise.} \end{cases} \tag{9}$$

Since the $\{\mathbf{E}_{ki}\}$ are circularly-symmetric complex Gaussian random matrices, $\{X_{ki}^{(mj)}, i = 1, \cdots, K, j = 1, \cdots, d\}$ are circularly-symmetric complex Gaussian random variables, and the left-hand side (LHS) of (8) is a *quadratic form of non-central Gaussian random variables*. To simplify notation, we will use vector form from here on. In vector form, (8) can be expressed as

$$\mathbf{X}_k^{(m)H} \mathbf{X}_k^{(m)} \geq \tau, \tag{10}$$

where $\mathbf{X}_k^{(m)} := [X_{k1}^{(m1)}, \cdots, X_{k1}^{(md)}, X_{k2}^{(m1)}, \cdots, X_{kK}^{(md)}]^T$. The elements of the mean vector $\boldsymbol{\mu}_k^{(m)} (:= \mathbb{E}\{\mathbf{X}_k^{(m)}\})$ of $\mathbf{X}_k^{(m)}$ are given by

$$[\boldsymbol{\mu}_k^{(m)}]_{(i-1)d+j} = \begin{cases} 0, & i = k, \ j = m, \\ \mathbf{u}_k^{(m)H} \hat{\mathbf{H}}_{ki} \mathbf{v}_i^{(j)}, & \text{otherwise,} \end{cases} \tag{11}$$

for $i = 1, \cdots, K$ and $j = 1, \cdots, d$, and the covariance matrix $\mathbf{\Sigma}_k^{(m)}$ of $\mathbf{X}_k^{(m)}$ is given by a block diagonal matrix, since $\{\mathbf{E}_{ki}, \ i = 1, \cdots, K\}$ are independent for different values of i, i.e.,

$$\begin{aligned} \mathbf{\Sigma}_k^{(m)} &:= \mathbb{E}\{(\mathbf{X}_k^{(m)} - \mathbb{E}\{\mathbf{X}_k^{(m)}\})(\mathbf{X}_k^{(m)} - \mathbb{E}\{\mathbf{X}_k^{(m)}\})^H\} \\ &= \text{diag}(\mathbf{\Sigma}_{k,1}^{(m)}, \cdots, \mathbf{\Sigma}_{k,K}^{(m)}), \end{aligned} \tag{12}$$

where the $d \times d$ sub-block matrix $\mathbf{\Sigma}_{k,i}^{(m)}$ is given by

$$\begin{aligned} \mathbf{\Sigma}_{k,i}^{(m)} =& \sigma_h^2 (\mathbf{u}_k^{(m)H} \mathbf{\Sigma}_r \mathbf{u}_k^{(m)}) \\ & \cdot \begin{bmatrix} \mathbf{v}_i^{(1)H} \mathbf{\Sigma}_t \mathbf{v}_i^{(1)} & \mathbf{v}_i^{(2)H} \mathbf{\Sigma}_t \mathbf{v}_i^{(1)} & \cdots & \mathbf{v}_i^{(d)H} \mathbf{\Sigma}_t \mathbf{v}_i^{(1)} \\ \mathbf{v}_i^{(1)H} \mathbf{\Sigma}_t \mathbf{v}_i^{(2)} & \mathbf{v}_i^{(2)H} \mathbf{\Sigma}_t \mathbf{v}_i^{(2)} & \cdots & \mathbf{v}_i^{(d)H} \mathbf{\Sigma}_t \mathbf{v}_i^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_i^{(1)H} \mathbf{\Sigma}_t \mathbf{v}_i^{(d)} & \mathbf{v}_i^{(2)H} \mathbf{\Sigma}_t \mathbf{v}_i^{(d)} & \cdots & \mathbf{v}_i^{(d)H} \mathbf{\Sigma}_t \mathbf{v}_i^{(d)} \end{bmatrix} \end{aligned} \tag{13}$$

for each i. (The proof of (13) is given in the appendix.) In the following sections, we will derive closed-form expressions for (7), investigate the behavior of the outage probability as a function of several parameters, and propose an outage-based beam design algorithm.

III. The Computation of the Outage Probability

In this section, we first derive a closed-form expression for the outage probability in the general case of the Kronecker CSI error model, and then consider special cases. After this, we examine the behavior of the outage probability as a function of several important system parameters based on the Chernoff bound.

[1]The circular symmetry of a random matrix of the form $\mathbf{AZB}$ with constant matrices $\mathbf{A}$ and $\mathbf{B}$ and a circularly-symmetric complex Gaussian matrix $\mathbf{Z}$ can easily be shown by a similar technique to that used in the appendix.

$$\mathsf{SINR}_k^{(m)}|_{\hat{\mathcal{H}}} = \frac{|(\mathbf{u}_k^{(m)})^H \hat{\mathbf{H}}_{kk} \mathbf{v}_k^{(m)}|^2}{|(\mathbf{u}_k^{(m)})^H \mathbf{E}_{kk} \mathbf{v}_k^{(m)}|^2 + \sum_{j \neq m} |(\mathbf{u}_k^{(m)})^H (\hat{\mathbf{H}}_{kk} + \mathbf{E}_{kk}) \mathbf{v}_k^{(j)}|^2 + \sum_{i \neq k} \sum_{j=1}^{d} |(\mathbf{u}_k^{(m)})^H (\hat{\mathbf{H}}_{ki} + \mathbf{E}_{ki}) \mathbf{v}_i^{(j)}|^2 + \sigma^2}. \tag{6}$$

A. Closed-form expressions for the outage probability

For a Gaussian random vector $\mathbf{X} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with the eigendecomposition of its covariance matrix $\boldsymbol{\Sigma} = \boldsymbol{\Psi}\boldsymbol{\Lambda}\boldsymbol{\Psi}^H$, the CDF of $\mathbf{X}^H \bar{\mathbf{Q}} \mathbf{X}$ for some given $\bar{\mathbf{Q}}$ is given by [25]

$$\Pr\{\mathbf{X}^H \bar{\mathbf{Q}} \mathbf{X} \leq \tau\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\tau(\iota\omega+\beta)}}{\iota\omega+\beta} \frac{e^{-c}}{\det(\mathbf{I} + (\iota\omega+\beta)\mathbf{Q})} d\omega \tag{14}$$

for some $\beta > 0$ such that $\mathbf{I} + \beta\mathbf{Q}$ is positive definite, where $\mathbf{Q} = \boldsymbol{\Lambda}^{H/2}\boldsymbol{\Psi}^H \bar{\mathbf{Q}} \boldsymbol{\Psi} \boldsymbol{\Lambda}^{1/2}$, $\boldsymbol{\chi} = \boldsymbol{\Lambda}^{-1/2}\boldsymbol{\Psi}^H \boldsymbol{\mu}$ and $c = \boldsymbol{\chi}^H \left(\mathbf{I} + \frac{1}{\iota\omega+\beta}\mathbf{Q}^{-1}\right)^{-1} \boldsymbol{\chi}$. From here on, we will derive closed-form series expressions for the CDF of the outage probability in several important cases by applying the residue theorem used in [22] to the integral form (14) for the CDF. First, we consider the most general case of the Kronecker CSI error model. The outage probability in this case is given by the following theorem.

Theorem 1: For given transmit and receive beamforming matrices $\{\mathbf{V}_k = [\mathbf{v}_k^{(1)}, \cdots, \mathbf{v}_k^{(d)}]\}$ and $\{\mathbf{U}_k = [\mathbf{u}_k^{(1)}, \cdots, \mathbf{u}_k^{(d)}]\}$ designed based on $\hat{\mathcal{H}} = \{\hat{\mathbf{H}}_{ki}, \boldsymbol{\Sigma}_t, \boldsymbol{\Sigma}_r\}$, the outage probability for stream m of user k with the target rate $R_k^{(m)}$ under the CSI error model (2) and (3) is given by (15) on the next page where τ is given in (8); $\{\lambda_i, i = 1, \cdots, \kappa\}$ are all the distinct eigenvalues of the $Kd \times Kd$ covariance matrix $\boldsymbol{\Sigma}_k^{(m)}$ in (12) with eigendecomposition $\boldsymbol{\Sigma}_k^{(m)} = \boldsymbol{\Psi}_k^{(m)} \boldsymbol{\Lambda}_k^{(m)} \boldsymbol{\Psi}_k^{(m)H}$; κ_i is the multiplicity[2] of the eigenvalue λ_i; $\chi_i^{(j)}$ is the element of vector

$$\boldsymbol{\chi}_k^{(m)} := (\boldsymbol{\Lambda}_k^{(m)})^{-\frac{1}{2}} \boldsymbol{\Psi}_k^{(m)H} \boldsymbol{\mu}_k^{(m)} \tag{16}$$

corresponding to the j-th eigenvector of the eigenvalue λ_i ($1 \leq j \leq \kappa_i$), i.e., it is the j-th element of $(\lambda_i \mathbf{I}_{\kappa_i})^{-\frac{1}{2}} \boldsymbol{\Psi}_{k,i}^{(m)H} \boldsymbol{\mu}_k^{(m)}$ ($\boldsymbol{\Psi}_{k,i}^{(m)}$ is a $Kd \times \kappa_i$ matrix composed of the eigenvectors of $\boldsymbol{\Sigma}_k^{(m)}$ associated with λ_i.);

$$g_i(s) = \frac{e^{\tau s}}{s - 1/\lambda_i} \cdot \frac{\exp\left(-\sum_{p \neq i} \frac{(s-1/\lambda_i)\lambda_p}{1+(s-1/\lambda_i)\lambda_p} \sum_{q=1}^{\kappa_p} |\chi_p^{(q)}|^2\right)}{\prod_{p \neq i} \left(1 + (s - 1/\lambda_i)\lambda_p\right)^{\kappa_p}}; \tag{17}$$

and $g_i^{(n)}(s)$ is the n-th derivative of $g_i(s)$ w.r.t. s.

Proof: By using (14) and the facts $\bar{\mathbf{Q}} = \mathbf{I}$ and $\mathbf{X}_k^{(m)} \sim \mathcal{CN}(\boldsymbol{\mu}_k^{(m)}, \boldsymbol{\Sigma}_k^{(m)})$ in this case, we obtain the outage probability for stream m of user k in an integral form as

$$\Pr\{\mathbf{X}_k^{(m)H} \mathbf{X}_k^{(m)} \geq \tau\} = 1 - \frac{1}{2\pi\iota} \int_{\beta-\iota\infty}^{\beta+\iota\infty} \frac{e^{s\tau}}{s} \cdot \frac{e^{-\sum_{i=1}^{\kappa} \frac{s\lambda_i}{1+s\lambda_i} (\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)}}{\prod_{i=1}^{\kappa} (1 + s\lambda_i)^{\kappa_i}} ds, \tag{18}$$

where $s = \beta + \iota\omega$ ($\beta > 0$). The outage probability (18) can be expressed as a contour integral:

$$\Pr\{\mathbf{X}_k^{(m)H} \mathbf{X}_k^{(m)} \geq \tau\} = 1 - \frac{1}{2\pi\iota} \oint_{\mathcal{C}} \underbrace{\frac{e^{s\tau}}{s} \cdot \frac{e^{-\sum_{i=1}^{\kappa} \frac{s\lambda_i}{1+s\lambda_i} (\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)}}{\prod_{i=1}^{\kappa} (1 + s\lambda_i)^{\kappa_i}}}_{=:F(s)} ds, \tag{19}$$

where $\mathcal{C}$ is a contour of integration containing the imaginary axis and the whole left half plane of the complex plane. By the residue theorem, the sum of the residues at singular points of $F(s)$ which do not have positive real parts yields the contour integral in (19) times $2\pi\iota$. It is easy to see that the singular points of $F(s)$ are $s = 0$ and $s = -1/\lambda_i$, $i = 1, \cdots, \kappa$. Since $\boldsymbol{\Sigma}_{k,i}^{(m)}$ are all positive-definite, $\boldsymbol{\Sigma}_k^{(m)}$ is positive definite and $\lambda_i > 0$ for all i. So, the outage probability is given by

$$\Pr\{\text{outage}\} = 1 - \Big(\operatorname*{Res}_{s=0} F(s) + \sum_{i=1}^{\kappa} \operatorname*{Res}_{s=-1/\lambda_i} F(s)\Big). \tag{20}$$

It is also easy to see from (19) that the residue of $F(s)$ at $s = 0$ is $\operatorname*{Res}_{s=0} F(s) = 1$. To compute $\operatorname*{Res}_{s=-1/\lambda_i} F(s)$, for each i we introduce $G_i(s)$ defined as (21) in the next page. Now, the residue of $F(s)$ at $s = -1/\lambda_i$ is transformed to that of $G_i(s)$ at $s = 0$. The Laurent series expansion of $f_i(s)$ and the Taylor series expansion of $g_i(s)$ at $s = 0$ are given respectively by

$$f_i(s) = \frac{1}{(\lambda_i s)^{\kappa_i}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}{\lambda_i s}\right)^n \tag{22}$$

and

$$g_i(s) = \sum_{n=0}^{\infty} \frac{1}{n!} g_i^{(n)}(0) s^n. \tag{23}$$

By multiplying the two series and computing the coefficient of $1/s$, we obtain the residue of $G_i(s)$ at $s = 0$ as

$$\operatorname*{Res}_{s=0} G_i(s) = \frac{e^{-(\frac{\tau}{\lambda_i} + \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)}}{\lambda_i^{\kappa_i}} \times \sum_{n=\kappa_i-1}^{\infty} \frac{1}{n!} g_i^{(n)}(0) \frac{1}{(n-\kappa_i+1)!} \left(\frac{\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}{\lambda_i}\right)^{n-\kappa_i+1} \tag{24}$$

for each i. Finally, substituting the residues into (20) yields (15). ■

To compute (15), we need to compute $\{\lambda_i\}$, $\{\chi_i^{(j)}\}$ and the higher order derivatives of $g_i(s)$. The first two terms are easy to compute since they are related to the mean vector of size Kd and the covariance matrix of size $Kd \times Kd$. Furthermore, the higher order derivatives of $g_i(s)$ can also be computed efficiently based on a recursion. (Please see [27].) Note that in the case in which the elements $\mathbf{H}_{ki}^{(w)}$ in (3) have different

[2] Since $\boldsymbol{\Sigma}_k^{(m)}$ is a normal matrix, we have $Kd = \sum_{i=1}^{\kappa} \kappa_i$.

$$\Pr\{\text{outage}\} = \Pr\{\log_2(1+\mathsf{SINR}_k^{(m)}) \le R_k^{(m)}\}$$
$$= -\sum_{i=1}^{\kappa} \frac{e^{-(\frac{\tau}{\lambda_i}+\sum_{j=1}^{\kappa_i}|\chi_i^{(j)}|^2)}}{\lambda_i^{\kappa_i}} \sum_{n=\kappa_i-1}^{\infty} \frac{1}{n!} g_i^{(n)}(0) \frac{1}{(n-\kappa_i+1)!} \left(\frac{\sum_{j=1}^{\kappa_i}|\chi_i^{(j)}|^2}{\lambda_i} \right)^{n-\kappa_i+1} \quad (15)$$

$$G_i(s) := F\left(s-\frac{1}{\lambda_i}\right) = \frac{e^{\tau(s-1/\lambda_i)}}{s-1/\lambda_i} \cdot \frac{e^{-\sum_{p=1}^{\kappa} \frac{\lambda_p(s-1/\lambda_i)}{1+\lambda_p(s-1/\lambda_i)} (\sum_{q=1}^{\kappa_p}|\chi_p^{(q)}|^2)}}{\prod_{p=1}^{\kappa}(1+\lambda_p(s-1/\lambda_i))^{\kappa_p}}$$
$$= \frac{e^{\tau(s-1/\lambda_i)}}{s-1/\lambda_i} \cdot \frac{e^{-\frac{\lambda_i s-1}{\lambda_i s}\sum_{j=1}^{\kappa_i}|\chi_i^{(j)}|^2}}{(\lambda_i s)^{\kappa_i}} \cdot \underbrace{\frac{e^{-\sum_{p\neq i} \frac{\lambda_p(s-1/\lambda_i)}{1+\lambda_p(s-1/\lambda_i)} \sum_{q=1}^{\kappa_p}|\chi_p^{(q)}|^2}}{\prod_{p\neq i}(1+\lambda_p(s-1/\lambda_i))^{\kappa_p}}}_{=:I_1}$$
$$= e^{-(\frac{\tau}{\lambda_i}+\sum_{j=1}^{\kappa_i}|\chi_i^{(j)}|^2)} \times \underbrace{\frac{e^{\frac{1}{\lambda_i s}\sum_{j=1}^{\kappa_i}|\chi_i^{(j)}|^2}}{(\lambda_i s)^{\kappa_i}}}_{=:f_i(s)} \times \underbrace{\left(\frac{e^{\tau s}}{s-1/\lambda_i} \times I_1\right)}_{=:g_i(s)}. \quad (21)$$

variances, (15) is still valid since the different variances only change the covariance matrix (12) and the outage expression depends on the covariance matrix (12) through $\{\lambda_i\}$ and $\{\chi_i^{(j)}\}$.

Next, we provide some useful corollaries to Theorem 1 regarding the outage probability in meaningful special cases. First, we consider the case in which a subset of channels are perfectly known at receiver k, i.e., $\mathbf{H}_{ki} = \hat{\mathbf{H}}_{ki}$ and $\mathbf{E}_{ki} = \mathbf{0}$ for some $i \in \mathcal{K}$. This corresponds to the case in which channel estimation or CSI feedback for some links is easier than that for other links. For example, the desired link channel may be easier to estimate than others. The outage probability in this case is given by the following corollary.

Corollary 1: When perfect CSI for some channel links including the desired link is available at receiver k, i.e., $\hat{\mathbf{H}}_{ki} = \mathbf{H}_{ki}$ for $i \in \Upsilon_k \subset \mathcal{K}$, the outage probability for stream m of user k is given by

$$\Pr\{\text{outage}\}$$
$$= \Pr\{\log_2(1+\mathsf{SINR}_k^{(m)}) \le R_k^{(m)}\} \quad (25)$$
$$= -\sum_{i=1}^{\kappa'} \left[\frac{e^{-(\frac{\tau'}{\lambda_i}+\sum_{j=1}^{\kappa_i}|\chi_i^{(j)}|^2)}}{\lambda_i^{\kappa_i}} \times \sum_{n=\kappa_i-1}^{\infty} \frac{1}{n!} g_{1,i}^{(n)}(0) \frac{1}{(n-\kappa_i+1)!} \left(\frac{\sum_{j=1}^{\kappa_i}|\chi_i^{(j)}|^2}{\lambda_i} \right)^{n-\kappa_i+1} \right]$$

where τ' is defined below; $\{\lambda_i, i=1,\cdots,\kappa'\}$ is the set of all the distinct eigenvalues of the covariance matrix (12); κ_i is the multiplicity of λ_i, satisfying $(K-|\Upsilon_k|)d = \sum_{i=1}^{\kappa'}\kappa_i$; $\chi_i^{(j)}$ is given in (16); and

$$g_{1,i}(s) = \frac{e^{\tau' s}}{s-1/\lambda_i} \cdot \frac{\exp\left(-\sum_{p\neq i} \frac{(s-1/\lambda_i)\lambda_p}{1+(s-1/\lambda_i)\lambda_p} \sum_{q=1}^{\kappa_p}|\chi_p^{(q)}|^2\right)}{\prod_{p\neq i}\left(1+(s-1/\lambda_i)\lambda_p\right)^{\kappa_p}}. \quad (26)$$

Proof: When CSI for some links including the desired link is perfect, the outage event at stream m of user k is given by (27) on the next page, since $\mathbf{E}_{ki} = \mathbf{0}$ for $i \in \Upsilon_k$. Thus, in this case the outage event is expressed in a quadratic form as follows:

$$\sum_{i\in\Upsilon_k^c}\sum_{j=1}^{d} X_{ki}^{(mj)H} X_{ki}^{(mj)} \ge \frac{|\mathbf{u}_k^{(m)H}\hat{\mathbf{H}}_{kk}\mathbf{v}_k^{(m)}|^2}{2^{R_k^{(m)}}-1} \quad (28)$$
$$- \sum_{i\in\Upsilon_k}\sum_{\substack{j=1,\\ j\neq m}}^{d} |\mathbf{u}_k^{(m)H}\hat{\mathbf{H}}_{ki}\mathbf{v}_i^{(j)}|^2 - \sum_{\substack{i\in\Upsilon_k,\\ i\neq k}} |\mathbf{u}_k^{(m)H}\hat{\mathbf{H}}_{ki}\mathbf{v}_i^{(m)}|^2 - \sigma^2 =: \tau'$$

and we have $X_{ki}^{(mj)} \equiv 0$ for all $i \in \Upsilon_k$ (See (9)). The size of $\mathbf{X}_k^{(m)}$ now reduces to $(K-|\Upsilon_k|)d$, and the size of the covariance matrix $\mathbf{\Sigma}_k^{(m)}$ is $(K-|\Upsilon_k|)d \times (K-|\Upsilon_k|)d$. With the new threshold τ', the same argument as that in Theorem 1 can be applied to yield the result. ■

Thus, when perfect CSI is available for some links, the order of the distribution is reduced under the same structure. Next, consider the specific beam design method of interference alignment and the corresponding outage probability, which can be obtained by Corollary 1 and is given in the following corollary.

Corollary 2: When the desired channel link is perfectly known (i.e. $k \in \Upsilon_k$) and $\{\mathbf{V}_k\}$ and $\{\mathbf{U}_k\}$ are designed under IA based on $\hat{\mathcal{H}}$, the outage probability for stream m of user k is given by

$$\Pr\{\text{outage}\} = -\sum_{i=1}^{\kappa'} \frac{1}{\lambda_i^{\kappa_i}} e^{-\frac{\tau'}{\lambda_i}} \frac{1}{(\kappa_i-1)!} g_{1,i}^{(\kappa_i-1)}(0). \quad (29)$$

Proof: First, express the random term in (28) as $\sum_{i\in\Upsilon_k^c}\sum_{j=1}^{d} X_{ki}^{(mj)H}X_{ki}^{(mj)} = (\mathbf{X}_k^{(m)})^H\mathbf{X}_k^{(m)}$. When the beam is designed under IA based on $\hat{\mathcal{H}}$, we have $\mathbb{E}\{\mathbf{X}_k^{(m)}\} = \mathbf{0}$ since $\mathbf{u}_k^{(m)H}\hat{\mathbf{H}}_{ki}\mathbf{v}_i^{(j)} = 0$ for all $i \in \mathcal{K}\backslash\{k\} \supset \Upsilon_k^c$, $j=1,\cdots,d$. (See (11).) Hence, $\boldsymbol{\chi}_k^{(m)} = \mathbf{0}$ and thus $\chi_i^{(j)} = 0$ for all i and j. (See (16).) Then, the terms in the infinite series in (25) are zero for all $n > \kappa_i - 1$ from the fact that $0^0 = 1$ and $0! = 1$, and the result follows. ■

The outage probability for single stream communication is given in Corollary 3.

$$\log_2\left(1+\frac{|\mathbf{u}_k^{(m)H}\hat{\mathbf{H}}_{kk}\mathbf{v}_k^{(m)}|^2}{\sum_{i\in\Upsilon_k}\sum_{\substack{j=1,\\ j\neq m}}^{d}|\mathbf{u}_k^{(m)H}\hat{\mathbf{H}}_{ki}\mathbf{v}_i^{(j)}|^2+\sum_{\substack{i\in\Upsilon_k,\\ i\neq k}}|\mathbf{u}_k^{(m)H}\hat{\mathbf{H}}_{ki}\mathbf{v}_i^{(m)}|^2+\sum_{i\in\Upsilon_k^c}\sum_{j=1}^{d}|\mathbf{u}_k^{(m)H}(\hat{\mathbf{H}}_{ki}+\mathbf{E}_{ki})\mathbf{v}_i^{(j)}|^2+\sigma^2}\right)\leq R_k^{(m)} \quad (27)$$

Corollary 3: When $d=1$ and all eigenvalues of $\mathbf{\Sigma}_k^{(m)}$ are distinct, the outage probability for user k is given by

$$\begin{aligned}&\Pr\{\text{outage}\}\\ &=\Pr\{\log_2(1+\mathsf{SINR}_k)\leq R_k\}\\ &=-\sum_{i=1}^{K}\frac{e^{-(|\chi_i|^2+\tau/\lambda_i)}}{\lambda_i}\sum_{n=0}^{\infty}\left(\frac{1}{n!}\right)^2\left(\frac{|\chi_i|^2}{\lambda_i}\right)^n g_i^{(n)}(0),\end{aligned} \quad (30)$$

where $g_i(s)$ in (17) reduces to

$$g_i(s)=\frac{e^{\tau s}}{s-1/\lambda_i}\cdot\frac{e^{-\sum_{p\neq i}\frac{\lambda_p(s-1/\lambda_i)}{1+\lambda_p(s-1/\lambda_i)}|\chi_p|^2}}{\prod_{p\neq i}\Big(1+\lambda_p(s-1/\lambda_i)\Big)}.$$

(Here, we have omitted the stream superscripts since the stream index is unique.)

Proof: Since all eigenvalues are assumed to be distinct, there are $\kappa=K$ eigenvalues with $\kappa_i=1$ for all i. Substituting these into Theorem 1 yields the result. ■

Now, let us consider a simpler case for $d=1$ with no antenna correlation. In this case, the outage probability is given as an explicit function of the channel uncertainty level σ_h^2, and it is given by the following corollary to Theorem 1.

Corollary 4: When $d=1$ and there is no antenna correlation, the outage probability is given by

$$\begin{aligned}\Pr\{\text{outage}\}&=-\frac{1}{(\sigma_h^2)^K}e^{-(\frac{\tau}{\sigma_h^2}+\|\boldsymbol{\chi}_k\|^2)}\\ &\times\sum_{n=K-1}^{\infty}\frac{1}{n!}g^{(n)}(0)\frac{1}{(n-K+1)!}\left(\frac{\|\boldsymbol{\chi}_k\|^2}{\sigma_h^2}\right)^{n-K+1},\end{aligned} \quad (31)$$

where $\boldsymbol{\chi}_k=\mathbb{E}\{\mathbf{X}_k\}/\sigma_h$ and $g(s)=\frac{e^{\tau s}}{s-1/\sigma_h^2}$.

Proof: In this case, an outage at user k occurs if and only if $\mathbf{X}_k^H\mathbf{X}_k\geq\frac{|\mathbf{u}_k^H\hat{\mathbf{H}}_{kk}\mathbf{v}_k|^2}{2^{R_k}-1}-\sigma^2$. Now, the covariance matrix $\mathbf{\Sigma}_k$ of $\mathbf{X}_k$ is $\sigma_h^2\mathbf{I}_K$ (see (12) and (13)), and thus there is only one eigenvalue σ_h^2 with multiplicity K. Moreover, $\boldsymbol{\chi}_k=\mathbb{E}\{\mathbf{X}_k\}/\sigma_h$ from (16) since $\mathbf{\Psi}_k=\mathbf{I}$ and $\mathbf{\Lambda}_k=\sigma_h^2\mathbf{I}$. By substituting these into Theorem 1, the outage probability (31) is obtained. ■

B. The behavior analysis of the outage probability based on the Chernoff bound

The obtained exact expressions for the outage probability in the previous subsection can easily be computed numerically, and will be used for the robust beam design based on the outage probability in Section IV. Before we address the outage-based robust beam design problem, let us investigate the behavior of the outage probability as a function of several parameters. Suppose that transmit and receive beam vectors $\{\mathbf{v}_k^{(m)},\mathbf{u}_k^{(m)}\}$ are designed by some known method based on $\hat{\mathcal{H}}$. For the given beam vectors, as seen in the obtained expressions, the outage probability is a function of other system parameters such as the known channel mean $\{\hat{\mathbf{H}}_{ki}\}$, the noise variance σ^2, the channel uncertainty level σ_h^2, the antenna correlation $\mathbf{\Sigma}_t$ and $\mathbf{\Sigma}_r$, and the target rate $R_k^{(m)}$. Here, the dependence on $\hat{\mathbf{H}}_{kk}$, σ^2 and $R_k^{(m)}$ is via the threshold $\tau(\hat{\mathbf{H}}_{kk},\sigma^2,R_k^{(m)})$, and the dependence on σ_h^2, $\mathbf{\Sigma}_t$, $\mathbf{\Sigma}_r$ and $\{\hat{\mathbf{H}}_{ki},i\neq k\}$ is via $\boldsymbol{\chi}_k^{(m)}(\mathbf{\Sigma}_k^{(m)}(\sigma_h^2,\mathbf{\Sigma}_t,\mathbf{\Sigma}_r),\mathbb{E}\{\mathbf{X}_k^{(m)}\}(\hat{\mathbf{H}}_{ki}))$ and the eigenvalues of $\mathbf{\Sigma}_{k,i}^{(m)}(\sigma_h^2,\mathbf{\Sigma}_t,\mathbf{\Sigma}_r)$. This complicated dependence structure makes it difficult to analyze the properties of the outage probability as a function of the system parameters. Thus, in this subsection we apply the Chernoff bounding technique [17] to the tractable[3] case of $d=1$ to obtain insights into the outage probability as a function of several important parameters. When $d=1$, the outage event is expressed as

$$\begin{aligned}\Pr\left\{\mathbf{X}_k^H\mathbf{X}_k\geq\tau=\frac{|\mathbf{u}_k^H\hat{\mathbf{H}}_{kk}\mathbf{v}_k|^2}{(2^{R_k}-1)}-\sigma^2\right\}\\ =\Pr\left\{\sum_{i=1}^{K}X_{ki}^HX_{ki}\geq\tau\right\}.\end{aligned} \quad (32)$$

Since $\mathbf{E}_{k1},\cdots,\mathbf{E}_{kK}$ are independent and circularly-symmetric complex Gaussian random matrices, $X_{k1},\cdots,X_{kK}$ are independent and circularly-symmetric complex Gaussian random variables. (See (9).) Thus, the term on the LHS in the second bracket in (32) is a sum of independent random variables, and the Chernoff bound can be applied to yield

$$\Pr\{\mathbf{X}_k^H\mathbf{X}_k\geq\tau\}\leq e^{-\tau s}\prod_{i=1}^{K}\mathbb{E}\left\{e^{s|X_{ki}|^2}\right\} \quad (33)$$

for any $s>0$. The moment generating function (m.g.f.) of $|X_{ki}|^2$ ($X_{ki}\sim\mathcal{CN}(\mu_{ki},\sigma_{ki}^2)$) is given by $\mathbb{E}\{e^{s|X_{ki}|^2}\}=\frac{1}{1-\sigma_{ki}^2s}\exp\left(\frac{|\mu_{ki}|^2s}{1-\sigma_{ki}^2s}\right)$ for $s<1/\sigma_{ki}^2$, where $\mu_{kk}=0$, $\mu_{ki}=\mathbf{u}_k^H\hat{\mathbf{H}}_{ki}\mathbf{v}_i$ for $i\neq k$, and $\sigma_{ki}^2=\sigma_h^2(\mathbf{u}_k^H\mathbf{\Sigma}_r\mathbf{u}_k)(\mathbf{v}_i^H\mathbf{\Sigma}_t\mathbf{v}_i)$. (See (9,11,13).) Therefore, the Chernoff bound on the outage probability is given by

$$\begin{aligned}&\Pr\{\mathbf{X}_k^H\mathbf{X}_k\geq\tau\}\\ &\leq e^{-\tau s}\prod_{i=1}^{K}\frac{1}{1-\sigma_{ki}^2s}\exp\left(\frac{|\mu_{ki}|^2s}{1-\sigma_{ki}^2s}\right)\\ &=\exp\left\{-\left[\tau s+\sum_{i=1}^{K}\log(1-\sigma_{ki}^2s)+\sum_{i=1}^{K}\frac{|\mu_{ki}|^2s}{\sigma_{ki}^2s-1}\right]\right\}\end{aligned} \quad (34)$$

[3]In certain cases of $d>1$, the Chernoff bound can still be obtained when each element in $\mathbf{X}_k^{(m)}$ is independent of the others. Such cases include the case in which there is no antenna correlation and the transmit beam vectors are orthogonal as in the IA beam case. In this case, similar results to the case of $d=1$ are obtained.

for $0 < s < \min_i\{1/\sigma_{ki}^2\}$. Now, (34) provides a tool to analyze the behavior of the outage probability as a function of several important parameters. The most desired property is the behavior of the outage probability as a function of the channel uncertainty level. This behavior is explained in the following theorem.

Theorem 2: When $d = 1$, as $\sigma_h^2 \to 0$, the outage probability decreases to zero, and the decay rate is given by

$$\Pr\{\text{outage}\} \le e^{-c_1} \cdot \exp(-c_2/\sigma_h^2) \tag{35}$$

for some c_1 and $c_2 > 0$ not depending on σ_h^2, if the target rate R_k and the designed transmit and receive beam vectors $\{\mathbf{v}_k, \mathbf{u}_k\}$ satisfy

$$R_k < \bar{R}_k = \log_2\left(1 + \frac{|\mathbf{u}_k^H \hat{\mathbf{H}}_{kk}\mathbf{v}_k|^2}{\sum_{i=1}^K \frac{|\mu_{ki}|^2}{1-\frac{(\mathbf{u}_k^H \mathbf{\Sigma}_r \mathbf{u}_k)(\mathbf{v}_i^H \mathbf{\Sigma}_t \mathbf{v}_i)}{tr(\mathbf{\Sigma}_r)tr(\mathbf{\Sigma}_t)}} + \sigma^2}\right). \tag{36}$$

Proof: (34) is valid for any $s \in (0, \min_i\{1/\sigma_{ki}^2\})$. So, let $s = 1/\sigma_h^2 \text{tr}(\mathbf{\Sigma}_t)\text{tr}(\mathbf{\Sigma}_r)$ $(< \min_i\{1/\sigma_{ki}^2\}$ since $||\mathbf{v}_k|| = ||\mathbf{u}_k|| = 1$ and $\sigma_{ki}^2 = \sigma_h^2(\mathbf{u}_k^H\mathbf{\Sigma}_r\mathbf{u}_k)(\mathbf{v}_i^H\mathbf{\Sigma}_t\mathbf{v}_i) \le \sigma_h^2\text{tr}(\mathbf{\Sigma}_t)\text{tr}(\mathbf{\Sigma}_r)$ for all i). Then, the exponent in (34) is given by (37) on the next page. Now, substituting $\tau = |\mathbf{u}_k^H\hat{\mathbf{H}}_{kk}\mathbf{v}_k|^2/(2^{R_k}-1) - \sigma^2$ into the inequality $c_2 > 0$ yields (36). ■

Theorem 2 states that the outage probability decays to zero as the CSI quality improves, more precisely, it decays exponentially w.r.t. the inverse of channel estimation MSE (or equivalently w.r.t. the channel K factor), if the target rate is below $\bar{R}_k$. In the Fisherian inference framework, the inverse of estimation MSE is information. Thus, another way we can view the above is that *the outage probability decays exponentially as the Fisher information for channel state increases, if the target rate is below a certain value.* So, the outage probability due to channel uncertainty is another case in which information is the error exponent as in many other inference problems. In certain cases, the condition (36) can be simplified considerably. For example, when interference-aligning beam vectors based on $\hat{\mathcal{H}}$ are used at the transmitters and receivers, we have $\mu_{ki} = \mathbf{u}_k^H\hat{\mathbf{H}}_{ki}\mathbf{v}_i = 0$ for $i \ne k$ in addition to $\mu_{kk} = 0$, and the condition is simplified to $R_k < \log_2\left(1 + \frac{|\mathbf{u}_k^H\hat{\mathbf{H}}_{kk}\mathbf{v}_k|^2}{\sigma^2}\right)$. Thus, in the case of interference alignment the outage probability can be made arbitrarily small by improving the CSI quality if the target rate is strictly less than the rate obtained by using $\hat{\mathbf{H}}_{kk}$ as the nominal channel. Next, consider the outage behavior as the effective SNR, $\Gamma_{eff} := |\mathbf{u}_k^H\hat{\mathbf{H}}_{kk}\mathbf{v}_k|^2/\sigma^2$, increases. Since the two terms determining the effective SNR are contained only in τ, it is straightforward to see from (34) that

$$\Pr\{\text{outage}\} \le c_3 \exp\left(-c_4\Gamma_{eff}\right), \tag{38}$$

for some c_3 and $c_4 = s\sigma^2/(2^{R_k}-1) > 0$ not depending on Γ_{eff}. Finally, consider the case in which the target rate R_k decreases. One can expect that the outage probability decays to zero if the target rate decreases to zero. The decaying behavior in this case is given in the following theorem.

Theorem 3: When $d = 1$, as $R_k \to 0$, the outage probability decreases to zero, and the decay rate is given by

$$\Pr\{\text{outage}\} \le c_6 \exp\left(-\frac{c_7}{2^{R_k}-1}\right) = c_6 \exp\left(-\frac{c_7'}{R_k + o(R_k)}\right) \tag{39}$$

for some $c_7,\ c_7' > 0$ not depending on R_k. The last equality is when R_k is near zero.

Proof: Let s be any positive constant contained in an interval $(0,\ 1/\max_i\{\sigma_h^2(\mathbf{u}_k^H\mathbf{\Sigma}_r\mathbf{u}_k)(\mathbf{v}_i^H\mathbf{\Sigma}_t\mathbf{v}_i)\})$. Then, the exponent in (34) becomes (40) on the next page. Hence, the Chernoff bound is given by $\Pr\{\text{outage}\} \le c_6\exp\left(-\frac{s|\mathbf{u}_k^H\hat{\mathbf{H}}_{kk}\mathbf{v}_k|^2}{2^{R_k}-1}\right) = c_6\exp\left(-\frac{c_7'}{R_k+o(R_k)}\right)$ for some $c_7' > 0$. The last equality is when R_k is near zero. In this case, we have $2^{R_k} - 1 = (\log 2)R_k + o(R_k)$ by Taylor's expansion. ■

IV. Outage-Based Robust Beam Design

In this section, we propose an outage-based beam design algorithm based on the closed-form expressions for the outage probability derived in the previous section. Our assumption is that $\hat{\mathcal{H}}$ is given for the beam design, as mentioned earlier. Suppose that transmit and receive beamforming matrices $\{\mathbf{V}_k, \mathbf{U}_k\}$ are designed by using any available beam design method based on $\hat{\mathcal{H}}$. Based on the designed $\{\mathbf{V}_k, \mathbf{U}_k\}$ and known $\{\hat{\mathcal{H}}, \sigma^2\}$, one can compute and use a nominal rate for transmission. Since $\hat{\mathcal{H}}$ is not perfect, however, an outage may occur depending on the CSI error if the nominal rate is used for transmission. Of course, the outage probability can be made small by making the transmission rate low or by improving the CSI quality, as seen in Section III-B. However, these methods are inefficient sometimes since we may have limitations in the CSI quality or need as high rate as possible for given $\hat{\mathcal{H}}$. Further, in many wireless systems the target outage probability for transmission is determined and the data transmission is performed under such an outage constraint. Thus, we here consider the beam design problem when the outage probability is given as a system parameter. In particular, we consider the following per-stream based beam design problem to maximize the sum ϵ-outage rate for given $\hat{\mathcal{H}}$:

$$\underset{\{\mathbf{v}_k^{(m)}\},\{\mathbf{u}_k^{(m)}\}}{\text{maximize}} \quad \sum_{k=1}^K \sum_{m=1}^d R_k^{(m)} \tag{41}$$

$$\text{subject to} \quad \Pr\{\log_2(1 + \text{SINR}_k^{(m)}|_{\hat{\mathcal{H}}}) \le R_k^{(m)}\} \le \epsilon \tag{42}$$

$$\|\mathbf{u}_k^{(m)}\| = \|\mathbf{v}_k^{(m)}\| = 1, \quad \forall k \in \mathcal{K},\ m = 1, \cdots, d, \tag{43}$$

where the ϵ-outage rate for stream m of user k is the maximum rate satisfying (42). Like other beam design problems in MIMO interference channels, the simultaneous joint optimal design for all transmit and receive beam vectors for this problem also seems difficult. Hence, we propose an iterative approach to the above sum ϵ-outage rate maximization problem. The proposed method is explained as follows. In the first step, we initialize $\{\mathbf{v}_k^{(m)}\}$ and $\{\mathbf{u}_k^{(m)}\}$ properly (here a known beam design algorithm for the MIMO interference channel can be used), and then find an optimal rate-tuple

$$
\begin{aligned}
& -\frac{\tau}{\sigma_h^2 \mathrm{tr}(\mathbf{\Sigma}_t)\mathrm{tr}(\mathbf{\Sigma}_r)} - \sum_{i=1}^{K} \log\left[1 - \frac{(\mathbf{u}_k^H \mathbf{\Sigma}_r \mathbf{u}_k)(\mathbf{v}_i^H \mathbf{\Sigma}_t \mathbf{v}_i)}{\mathrm{tr}(\mathbf{\Sigma}_t)\mathrm{tr}(\mathbf{\Sigma}_r)}\right] - \sum_{i=1}^{K} \frac{|\mu_{ki}|^2}{\sigma_h^2 (\mathbf{u}_k^H \mathbf{\Sigma}_r \mathbf{u}_k)(\mathbf{v}_i^H \mathbf{\Sigma}_t \mathbf{v}_i) - \sigma_h^2 \mathrm{tr}(\mathbf{\Sigma}_t)\mathrm{tr}(\mathbf{\Sigma}_r)} \\
= & -\frac{1}{\sigma_h^2} \underbrace{\left\{ \frac{\tau}{\mathrm{tr}(\mathbf{\Sigma}_t)\mathrm{tr}(\mathbf{\Sigma}_r)} + \sum_{i=1}^{K} \frac{|\mu_{ki}|^2}{(\mathbf{u}_k^H \mathbf{\Sigma}_r \mathbf{u}_k)(\mathbf{v}_i^H \mathbf{\Sigma}_t \mathbf{v}_i) - \mathrm{tr}(\mathbf{\Sigma}_t)\mathrm{tr}(\mathbf{\Sigma}_r)} \right\}}_{(=:c_2)} - \underbrace{\sum_{i=1}^{K} \log\left[1 - \frac{(\mathbf{u}_k^H \mathbf{\Sigma}_r \mathbf{u}_k)(\mathbf{v}_i^H \mathbf{\Sigma}_t \mathbf{v}_i)}{\mathrm{tr}(\mathbf{\Sigma}_t)\mathrm{tr}(\mathbf{\Sigma}_r)}\right]}_{(=:c_1)} \quad (37)
\end{aligned}
$$

$$
\begin{aligned}
& -\tau s - \underbrace{\sum_{i=1}^{K} \log[1 - \sigma_h^2 (\mathbf{u}_k^H \mathbf{\Sigma}_r \mathbf{u}_k)(\mathbf{v}_i^H \mathbf{\Sigma}_t \mathbf{v}_i) s] - \sum_{i=1}^{K} \frac{|\mu_{ki}|^2 s}{s\sigma_h^2 (\mathbf{u}_k^H \mathbf{\Sigma}_r \mathbf{u}_k)(\mathbf{v}_i^H \mathbf{\Sigma}_t \mathbf{v}_i) - 1}}_{(=:c_5)} \\
= & -\left(\frac{|\mathbf{u}_k^H \hat{\mathbf{H}}_{kk} \mathbf{v}_k|^2}{2^{R_k} - 1} - \sigma^2 \right) s - c_5 = -\frac{|\mathbf{u}_k^H \hat{\mathbf{H}}_{kk} \mathbf{v}_k|^2}{2^{R_k} - 1} s - c_5' \quad (40)
\end{aligned}
$$

$(R_1^{(1)}, \cdots, R_1^{(d)}, R_2^{(1)}, \cdots, R_K^{(d)})$ that maximizes the sum for given $\{\mathbf{v}_k^{(m)}, \mathbf{u}_k^{(m)}\}$ under the outage constraint. This step is performed based on the derived outage probability expressions in the previous section. Since designing each $R_k^{(m)}$ does not affect others, this step can be done separately for each $R_k^{(m)}$. Since the outage probability for stream m of user k increases monotonically w.r.t. $R_k^{(m)}$, the optimal $R_k^{(m)}$ in this step is the rate with the outage probability ϵ. In the second step, for the obtained rate-tuple and receive beam vectors $\{\mathbf{u}_k^{(m)}\}$ in the first step, we update the transmit beam vectors $\{\mathbf{v}_k^{(m)}\}$ to minimize the maximum of the outage probabilities of all streams and all users. (Since the outage probabilities of all streams of all users are ϵ at the end of the first step, this means that the outage probability decreases for all streams and all users.) Here, we apply the alternating minimization technique [29] to circumvent the difficulty in the joint transmit beam design. (The change in one transmit beam vector affects the outage probabilities of other users.) That is, we optimize one transmit beam vector while fixing all the others at a time. We iterate this procedure from the first stream of transmitter 1 to the last stream of user K until this step converge. In the third step, we design the receive beam vector $\mathbf{u}_k^{(m)}$ to minimize the outage probability at stream m of user k with the rate-tuple determined in the first step and $\{\mathbf{v}_k^{(m)}\}$ determined in the second step for each (k, m). This optimization can also be performed separately for each stream of each user since the receiver filter for one stream does not affect the performance of other streams. Finally, we go back to the first step with the updated transmit and receive beam vectors (in the revisited first step, the rate for each stream will be increased by increasing the outage probability up to ϵ again), and iterate the procedure until the sum ϵ-outage rate does not change. We have summarized the sum outage rate maximizing beam design algorithm in Table I.

Theorem 4: The proposed beam design algorithm converges.

Proof: It is straightforward to see that the sum ϵ-outage rate increases monotonically for each iteration of the three steps of the proposed algorithm. Also, the maximum sum rate is bounded by the rate with perfect CSI. Hence, the algorithm converges by the monotone convergence theorem for real sequences. ∎

TABLE I
THE PROPOSED ALGORITHM FOR SUM ϵ-OUTAGE RATE MAXIMIZATION WITH CHANNEL UNCERTAINTY

The Proposed Algorithm

Input: channel state estimate $\hat{\mathcal{H}}$ and allowed outage probability ϵ.

0. Initialize $\{\mathbf{v}_k^{(m)}\}$ and $\{\mathbf{u}_k^{(m)}\}$ as sets of unit-norm vectors properly.
1. For given $\{\mathbf{V}_k\}$ and $\{\mathbf{U}_k\}$, find $(R_1^{(1)}, \cdots, R_K^{(d)})$ that maximizes $\sum_{k=1}^{K} \sum_{m=1}^{d} R_k^{(m)}$ while the outage constraint is satisfied.
2. Update $\{\mathbf{V}_k = [\mathbf{v}_k^{(1)}, \cdots, \mathbf{v}_k^{(d)}]\}$ for $\{R_k^{(m)}\}$ and $\{\mathbf{U}_k^{(m)}\}$ given from step 1.
   - For pair (i, j), fix $\{\mathbf{v}_k^{(m)}, k = 1, \cdots, K, \; m = 1, \cdots, d\} \backslash \{\mathbf{v}_i^{(j)}\}$ and $\{\mathbf{U}_k\}$ and solve
   $$\mathbf{v}_i^{(j)} = \arg\min_{\mathbf{v} \in \mathbb{C}^{N_t}} \max_{k,m} \Pr\{\text{outage}_k^{(m)}\}. \quad (44)$$
   (Here, a commercial tool such as the MATLAB fminimax function can be used to solve (44) together with the derived outage expression.)
   - Iterate the above step from the first stream of transmitter 1 to the last stream of transmitter K until $\{\mathbf{V}_1, \cdots, \mathbf{V}_K\}$ converges.
3. For each receiver 1 to K, obtain the receive filter $\mathbf{u}_k^{(m)}$ that minimize the outage probability of stream m of receiver k for given $\{\mathbf{V}_k\}$ from step 2 and given $R_k^{(m)}$ from step 1. (Here, again a commercial tool such as the MATLAB fmincon function can be used together with the derived outage expression.)
4. Go to step 1 and repeat the whole procedure until the algorithm converges.

V. Numerical Results

In this section, we provide some numerical results to validate our series derivation, to examine the outage probability

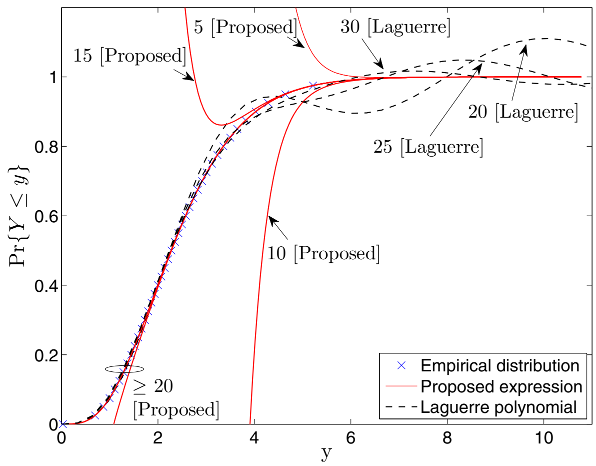


Fig. 1. Comparison of two series expressions for the CDF of a quadratic form of Gaussian random variables. $\mathbf{X} \sim \mathcal{CN}([0.5, 0.5, 0.5, 0.5]^T, 0.3\mathbf{I}_4)$, $\bar{\mathbf{Q}} = [1, 0.5, 0, 0; 0.5, 1, 0, 0; 0, 0, 1, 0; 0, 0, 0, 1]$, and $\beta = 2$ for Laguerre series expansion.

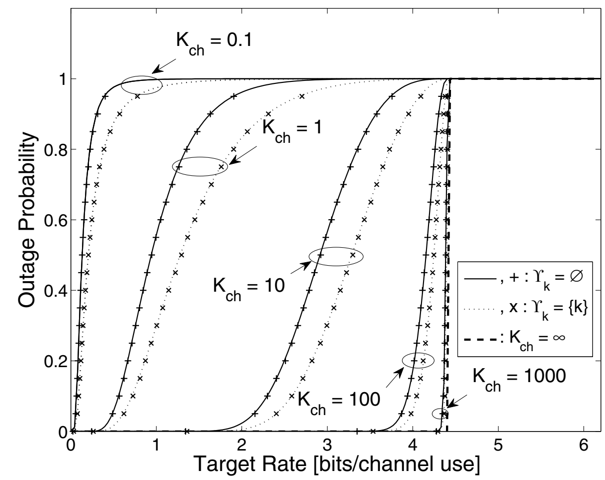


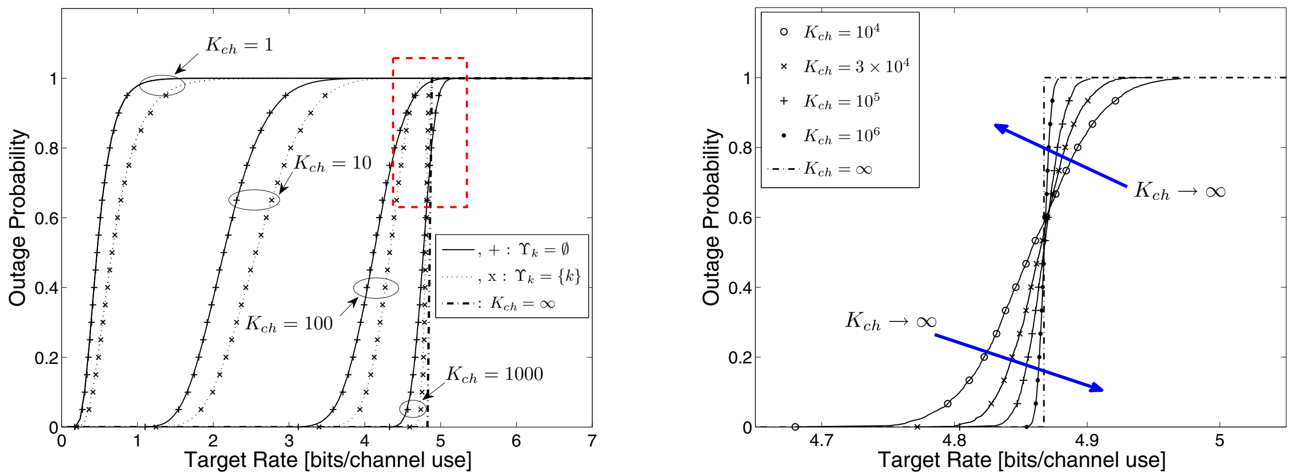
Fig. 2. Outage probability versus the target rate R_k ($K = 3$, $N_t = N_r = 2d = 2$, $\mathbf{\Sigma}_t = \mathbf{\Sigma}_r = \mathbf{I}$, $\Gamma = 15$ dB. Transmit and receive beam vectors are obtained by the IIA algorithm in [3].)

as a function of several system parameters and to evaluate the performance of the proposed beam design algorithm. For given $\mathbf{\Sigma}_t$, $\mathbf{\Sigma}_r$, $K_{ch}^{(ki)}$ and $\Gamma^{(k)}$, we first generated $\{\hat{\mathbf{H}}_{ki}\}$ randomly according to a zero-mean Gaussian distribution, and then scaled $\hat{\mathbf{H}}_{ki}$ to yield $\|\hat{\mathbf{H}}_{ki}\|_F^2 = N_t N_r$ for all (k, i). In this way, the channel K factor and the SNR were simply controlled by σ_h^2 and σ^2, respectively. After $\{\hat{\mathbf{H}}_{ki}\}$ were generated as such, we generated $\{\mathbf{E}_{ki}\}$ according to (3) and the true channel was determined by (2) if necessary[4]. For simplicity, we used $K_{ch}^{(ki)} = K_{ch}$ for all (k, i) and $\Gamma^{(k)} = \Gamma$ for all k.

First, Fig. 1 compares the convergence behavior of the derived series in this paper with that of the series fitting method [20], [21], [23], [26] based on the Laguerre basis functions for a given set of parameters shown in the label of the figure. It is seen that indeed our series converges from the upper tail first whereas the series fitting method converges from the lower tail first. (For a proof of this in the identity covariance matrix case, please refer to [27].) Note that the series fitting method yields large error at the upper tail distribution even with a reasonably large number of terms. With this verification, next consider the outage behavior as a function of several system parameters. Fig. 2 shows the outage probability w.r.t. the target rate R_k for a given set $\{\hat{\mathbf{H}}_{ki}\}$ (randomly generated as above) with several different channel K factors, when $K = 3, N_t = N_r = 2d = 2$, $\mathbf{\Sigma}_t = \mathbf{\Sigma}_r = \mathbf{I}$, $\Gamma = 15$ dB and the transmit and receive beam vectors were designed by the iterative interference alignment (IIA) algorithm [3]. The solid and dotted lines represent the result of our analysis, and the markers $+$ and $\times$ indicate the result of Monte Carlo runs for the outage probability. The theoretical outage curves in Fig. 2 were obtained by using (25) with the first 38 terms in the infinite series. It is seen that our analysis matches the results of the Monte Carlo runs very well. The dashed line shows the outage performance when $K_{ch} = \infty$, i.e., all transmitters and receivers have perfect CSI. In the case of $K_{ch} = \infty$, we have a sharp transition behavior across R_{limit} determined by the SINR (6) with $\mathbf{E}_{ki} = \mathbf{0}$ for all (k, i). It is seen that the outage performance deteriorates from the ideal step curve of $K_{ch} = \infty$, as the CSI quality degrades. The solid lines correspond to the outage performance for the finite values of K_{ch}, when the CSI for all channel links is imperfect. It is seen that $K_{ch} = 100$ (20 dB) yields reasonable outage performance compared with the perfect CSI case in this setup. Note that the gain in the outage probability by knowing the desired link perfectly is not negligible. (See the dotted lines.) Fig. 3 show the outage probability w.r.t. the target rate R_k for a given set $\{\hat{\mathbf{H}}_{ki}\}$ with several different K_{ch}, when $K = 3, N_t = N_r = 2d = 4$, $\mathbf{\Sigma}_t = \mathbf{\Sigma}_r = \mathbf{I}$, $\Gamma = 25$ dB and the transmit and receive beam vectors were designed by the IIA algorithm. Similar behavior is seen as in the single stream case, i.e., the outage performance generally deteriorates as K_{ch} decreases. However, it is interesting to observe in the multiple stream case that sufficiently good but not perfect CSI quality yields better outage performance than does perfect CSI in the high outage probability regime. (See Fig. 3 (b).) This implies that in the multiple stream case the second term (i.e., the self inter-stream interference term) in the denominator of the SINR formula (6) is made smaller by $\mathbf{E}_{kk}$'s being negatively aligned with $\mathbf{H}_{kk}$ than in the case of $\mathbf{E}_{kk} \equiv 0$. However, this is not useful in system operation since the system is operated in the low outage probability regime. All the theoretical curves in Figures 3 (a) and (b) were obtained from (25) with the first 45 terms in the infinite series. Fig. 4 shows the outage probability curves when the transmit and receive beamforming vectors are respectively chosen as the right and left singular vectors corresponding to the largest singular value of the desired channel and the other parameters are identical to the case in Fig. 2. A similar outage probability behavior to the previous case is observed.

Next, the outage probability w.r.t. the channel K factor for a given set $\{\hat{\mathbf{H}}\}$ for several values of the target rate R_k is

[4]The computation of the closed-form outage probability requires only the channel statistics and $\{\hat{\mathbf{H}}_{ki}\}$ regarding the channel information, but for Monte Carlo runs we need to generate $\{\mathbf{E}_{ki}\}$.



(a) (b)

Fig. 3. Outage probability versus the target rate R_k ($K = 3$, $N_t = N_r = 2d = 4$, $\mathbf{\Sigma}_t = \mathbf{\Sigma}_r = \mathbf{I}$, $\Gamma = 25$ dB. Transmit and receive beam vectors are designed by the IIA algorithm in [3].)

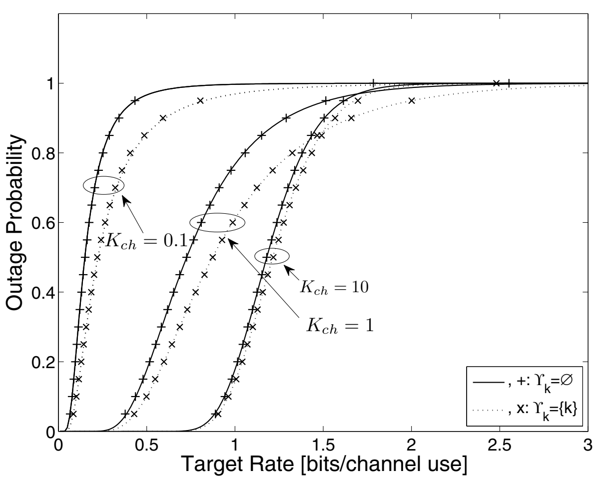


Fig. 4. Outage probability versus the target rate R_k ($K = 3$, $N_t = N_r = 2d = 2$, $\mathbf{\Sigma}_t = \mathbf{\Sigma}_r = \mathbf{I}$, $\Gamma = 15$ dB. Transmit and receive beam vectors are respectively chosen as the right and left singular vectors corresponding to the largest singular value of the desired channel matrix.)

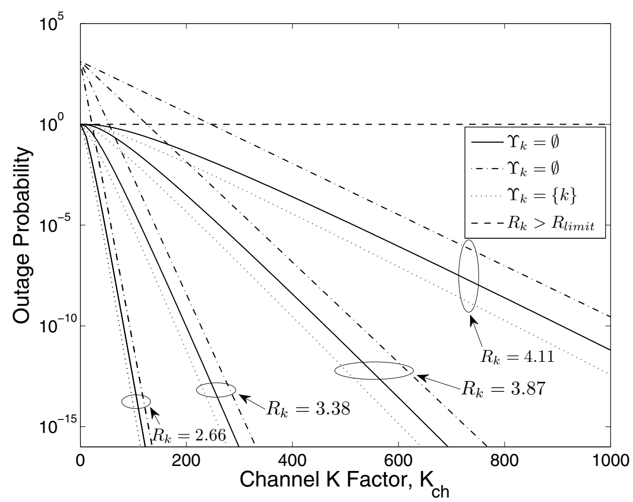


Fig. 5. Outage probability versus K_{ch} ($K = 3$, $N_t = N_r = 2d = 2$, $\mathbf{\Sigma}_t = \mathbf{\Sigma}_r = \mathbf{I}$, $\Gamma = 15$ dB. Transmit and receive beam vectors are designed by the IIA algorithm in [3].)

shown in Fig. 5, where the outage probability along the y-axis is drawn in log scale. (The same setup as for Fig. 2 was used and the IIA algorithm is used for the transmit and receive beam design. Here, (25) with the first 38 terms in the infinite series was used to compute the analytic curves.) As predicted by Theorem 2, the outage probability indeed decays exponentially w.r.t. the channel K factor (equivalently, w.r.t. the inverse of σ_h^2). The exponent depends on the target rate R_k; the higher the target rate is, the smaller the exponent is. This decaying behavior is also predicted in Theorem 2; the exponent c_2 in (35) is proportional to τ, and τ is inversely proportional to the target rate R_k. It is seen that the outage probability does not decay as K_{ch} increases, if R_k is larger than R_{limit}. In addition to the exact outage probability, the Chernoff bound in this case is shown in Fig. 5 as the lines with dots and dashes. It is seen that the Chernoff bound is not very tight but the decaying slope is the same as that of the exact outage probability.

Figures 6 and 7 show the impact of antenna correlation on the outage probability. We adopted the exponential antenna correlation profile considered in [30] and [31]. Under this model, the (i, j)-th element of the antenna correlation matrix $\mathbf{\Sigma}_t$ (or $\mathbf{\Sigma}_r$) in (3) is given by $\rho^{|i-j|}$, where $\rho \in [0, 1]$ is a parameter determining the correlation strength. Since $\text{tr}(\mathbf{\Sigma}_t) = N_t$ and $\text{tr}(\mathbf{\Sigma}_r) = N_r$ for this exponential antenna correlation model, we have the same transmit and receive powers as in the case of no antenna correlation, i.e., $\mathbf{\Sigma}_t = \mathbf{I}$ and $\mathbf{\Sigma}_r = \mathbf{I}$. Since the outage probability depends on $\{\hat{\mathbf{H}}_{ki}\}$ as well as on $\mathbf{\Sigma}_t$ and $\mathbf{\Sigma}_r$, we generated one hundred $\{\hat{\mathbf{H}}_{ki}\}$ randomly in the way that we explained already, and averaged the corresponding 100 outage probabilities to see the impact

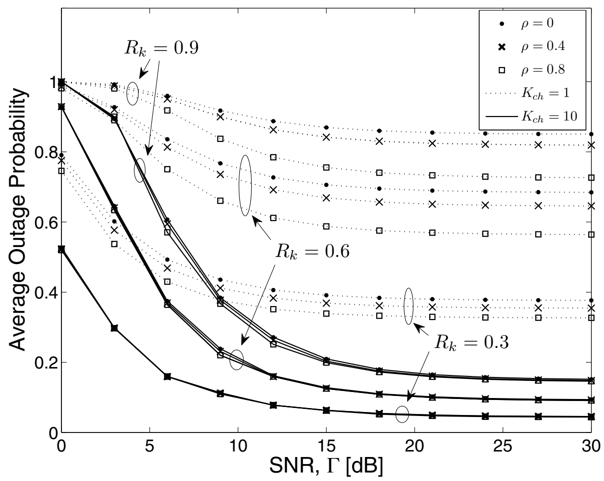


Fig. 6. Average outage probability versus Γ ($K = 3$, $N_t = N_r = 2d = 2$. Transmit and receive beam vectors designed by the IIA algorithm in [3].)

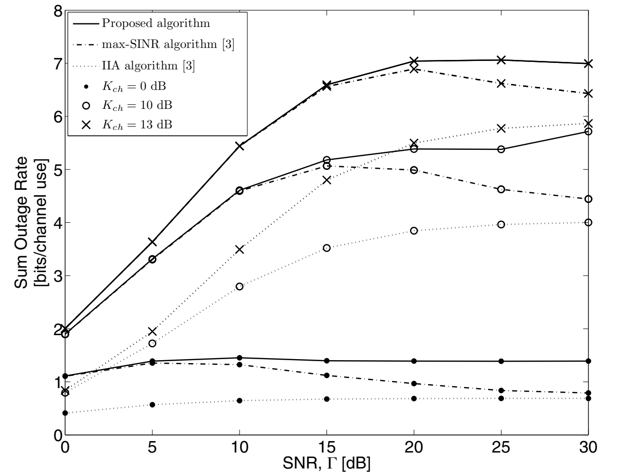


Fig. 8. Sum ϵ-outage rate for $\epsilon = 0.1$ ($K = 3$, $N_t = N_r = 2d = 2$, $\mathbf{\Sigma}_t = \mathbf{\Sigma}_r = \mathbf{I}$)

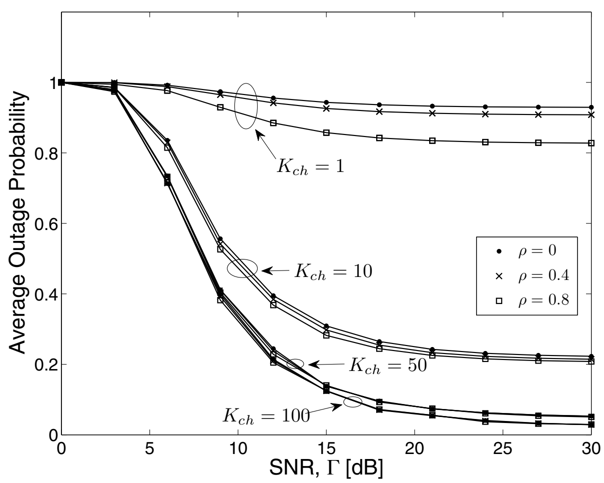


Fig. 7. Average outage probability versus Γ ($K = 3$, $N_t = N_r = 2d = 2$, $R_k = 1.2$. Transmit and receive beam vectors designed by the IIA algorithm in [3].)

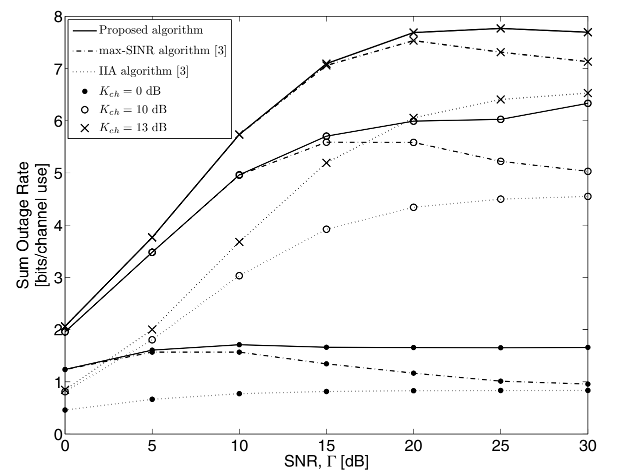


Fig. 9. Sum ϵ-outage rate for $\epsilon = 0.2$ ($K = 3$, $N_t = N_r = 2d = 2$, $\mathbf{\Sigma}_t = \mathbf{\Sigma}_r = \mathbf{I}$)

of the error correlation only. Other aspects of the system configuration were the same as those for Figures 2 and 5. It is seen that the error correlation decreases the outage probability especially when the CSI quality is very bad, but the gain becomes negligible when the CSI quality is good.

Finally, the performance of the proposed beam design algorithm maximizing the sum ϵ-outage rate was evaluated. As a reference, we adopted the max-SINR algorithm and IIA algorithm in [3]. Although the max-SINR and IIA algorithms were originally proposed to design beam vectors with perfect channel information, we applied the algorithms to design beam vectors by treating the imperfect channel $\hat{\mathcal{H}}$ as the true channel. The ϵ-outage rate of the max-SINR algorithm (or the IIA algorithm) is defined as the maximum rate that can be achieved under the outage constraint of ϵ using the beam vectors designed by the max-SINR algorithm (or the IIA algorithm). Once $\{\mathbf{V}_k\}$ and $\{\mathbf{U}_k\}$ are designed by any design method for given $\mathbf{\Sigma}_t$, $\mathbf{\Sigma}_r$ and $\{\hat{\mathbf{H}}_{ki}\}$, the outage probability corresponding to the designed beam vectors is easily computed as a function of the target rate R_k from Theorem 1. Thus, for the beam vectors designed by the max-SINR and IIA algorithms as well as for those designed by the proposed design algorithm in Section IV, the ϵ-outage rate R_k can easily be obtained. Figures 8 and 9 show the sum ϵ-outage rate of the proposed beam design method averaged over thirty different sets of $\{\hat{\mathbf{H}}_{ki}\}$ for $\epsilon = 0.1$ and $\epsilon = 0.2$, respectively, when $K = 3, N_t = N_r = 2d = 2$ and $\mathbf{\Sigma}_t = \mathbf{\Sigma}_r = \mathbf{I}$ for different K_{ch}'s. (The outage probability expression (31) with the first 40 terms was used to compute the outage probability.) It is seen that the proposed algorithm outperforms the IIA and max-SINR algorithms for all SNR, and the max-SINR algorithm shows good performance almost comparable to the proposed algorithm at low SNR. However, as SNR increases, the performance of the max-SINR algorithm degrades to that of the IIA algorithm (the two algorithm themselves converge as SNR increases) and there is a considerable gain by exploiting the channel uncertainty.

VI. Conclusion

In this paper, we have considered the outage probability and outage-based beam design for MIMO interference channels. We have derived closed-form expressions for the outage probability in MIMO interference channels under the assumption of Gaussian-distributed CSI error, and have derived the asymptotic behavior of the outage probability as a function of several system parameters based on the Chernoff bound. We have shown that the outage probability decreases exponentially w.r.t. the channel K factor defined as the ratio of the power of the known channel part and that of the unknown channel part. We have also provided an iterative beam design algorithm for maximizing the sum outage rate based on the derived outage probability expressions. Numerical results show that the proposed beam design method significantly outperforms conventional methods assuming perfect CSI in the sum outage rate performance.

Appendix

Proof of (13)*:* The (p,q)-th element of $\mathbf{\Sigma}_{k,i}^{(m)}$ is given by

$$\begin{aligned}
&\mathbb{E}\{(X_{ki}^{(mp)}-\mathbb{E}\{X_{ki}^{(mp)}\})(X_{ki}^{(mq)}-\mathbb{E}\{X_{ki}^{(mq)}\})^H\}\\
&=\mathbb{E}\{(\mathbf{u}_k^{(m)H}\mathbf{E}_{ki}\mathbf{v}_i^{(p)})(\mathbf{u}_k^{(m)H}\mathbf{E}_{ki}\mathbf{v}_i^{(q)})^H\}\\
&\overset{(a)}{=}\mathbb{E}\{(\mathbf{v}_i^{(p)T}\otimes\mathbf{u}_k^{(m)H})\text{vec}(\mathbf{E}_{ki})\text{vec}(\mathbf{E}_{ki})^H(\mathbf{v}_i^{(q)T}\otimes\mathbf{u}_k^{(m)H})^H\}\\
&\overset{(b)}{=}\sigma_h^2(\mathbf{v}_i^{(p)T}\otimes\mathbf{u}_k^{(m)H})(\mathbf{\Sigma}_t^T\otimes\mathbf{\Sigma}_r)(\mathbf{v}_i^{(q)T}\otimes\mathbf{u}_k^{(m)H})^H\\
&\overset{(c)}{=}\sigma_h^2(\mathbf{v}_i^{(p)T}\mathbf{\Sigma}_t^T\otimes\mathbf{u}_k^{(m)H}\mathbf{\Sigma}_r)(\mathbf{v}_i^{(q)*}\otimes\mathbf{u}_k^{(m)}),\\
&\qquad(\text{where}\quad \mathbf{v}_i^{(q)*}=(\mathbf{v}_i^{(q)T})^H)\\
&\overset{(d)}{=}\sigma_h^2(\mathbf{v}_i^{(p)T}\mathbf{\Sigma}_t^T\mathbf{v}_i^{(q)*}\otimes\mathbf{u}_k^{(m)H}\mathbf{\Sigma}_r\mathbf{u}_k^{(m)})\\
&\overset{(e)}{=}\sigma_h^2(\mathbf{v}_i^{(q)H}\mathbf{\Sigma}_t\mathbf{v}_i^{(p)})(\mathbf{u}_k^{(m)H}\mathbf{\Sigma}_r\mathbf{u}_k^{(m)}).
\end{aligned}$$

Here, (a) is obtained by applying $\text{vec}(\mathbf{ABC})=(\mathbf{C}^T\otimes\mathbf{A})\text{vec}(\mathbf{B})$ to each of the two terms in the expectation, (b) is by $\mathbb{E}\{\text{vec}(\mathbf{E}_{ki})\text{vec}(\mathbf{E}_{ki})^H\}=\sigma_h^2(\mathbf{\Sigma}_t^T\otimes\mathbf{\Sigma}_r)$, (c) and (d) are by $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D})=(\mathbf{AC}\otimes\mathbf{BD})$, and finally (e) is because $\mathbf{v}_i^{(p)T}\mathbf{\Sigma}_t^T\mathbf{v}_i^{(q)*}$ and $\mathbf{u}_k^{(m)H}\mathbf{\Sigma}_r\mathbf{u}_k^{(m)}$ are scalars. ∎

References

[1] J. Park, D. Kim, and Y. Sung, "Sum outage-rate maximization for MIMO interference channels," in *Proc. 2011 IEEE Global Communications Conf.*

[2] V. R. Cadambe and S. A. Jafar, "Interference alignment and degrees of freedom of the K-user interference channel," *IEEE Trans. Inf. Theory*, vol. 54, pp. 3425–3441, Aug. 2008.

[3] K. Gomadam, V. R. Cadambe, and S. A. Jafar, "A distributed numerical approach to interference alignment and applications to wireless interference networks," *IEEE Trans. Inf. Theory*, vol. 57, pp. 3309–3322, June 2011.

[4] S. W. Peters and R. W. Heath, "Cooperative algorithms for MIMO interference channels," *IEEE Trans. Veh. Technol.*, vol. 60, pp. 206–218, Jan. 2011.

[5] H. Yu and Y. Sung, "Least squares approach to joint beam design for interference alignment in multiuser multi-input multi-output interference channels," *IEEE Trans. Signal Process.*, vol. 58, pp. 4960–4966, Sep. 2010.

[6] I. Santamaria, O. Gonzalez, R. W. Heath, and S. W. Peters, "Maximum sum-rate interference alignment algorithms for MIMO channels," in *Proc. 2010 IEEE Global Communications Conf.*

[7] D. Schmidt, S. Changxin, R. Berry, M. Honig, and W. Utschick, "Minimum mean squared error interference alignment," in *Proc. 2009 Asilomar Conf. on Signals, Systems and Computers*.

[8] F. Negro, S. P. Shenoy, I. Ghauri, and D. T. M. Slock, "Weighted sum rate maximization in the MIMO interference channel," in *Proc. 2010 IEEE Int. Symp. on Personal Indoor and Mobile Radio Commun.*

[9] J. Shin and J. Moon, "Weighted sum rate maximizing transceiver design in MIMO interference channel," in *Proc. 2011 IEEE Global Communications Conf.*

[10] E. Biglieri *et al.*, *MIMO Wireless Communications*. Cambridge University Press, 2007.

[11] J. Lindblom, E. Karipidis, and E. Larsson, "Outage rate regions for the MISO interference channel: definitions and interpretations," ArXiv pre-prints cs.IT/1106.5615v1, June 2011.

[12] J. Lindblom, E. Larsson, and E. Jorswieck, "Parametrization of the MISO IFC rate region: the case of partial channel state information," *IEEE Trans. Wireless Commun.*, vol. 9, pp. 500–504, Feb. 2010.

[13] W.-C. Chiang, T.-H. Chang, C. Lin, and C.-Y. Chi, "Coordinated beamforming for multiuser MISO interference channel under rate outage constraints," ArXiv pre-prints cs.IT/1108.4475v1, Aug. 2011.

[14] E. Chiu, V. Lau, H. Huang, T. Wu, and S. Liu, "Robust transceiver design for K-pairs quasi-static MIMO interference channels via semi-definite relaxation," *IEEE Trans. Wireless Commun.*, vol. 9, pp. 3762–3769, Dec. 2010.

[15] H. Shen, B. Li, M. Tao, and X. Wang, "MSE-based transceiver designs for the MIMO interference channel," *IEEE Trans. Wireless Commun.*, vol. 9, pp. 3480–3489, Sep. 2010.

[16] S. Ghosh, B. Rao, and J. Zeidler, "Outage-efficient strategies for multiuser MIMO networks with channel distribution information," *IEEE Trans. Signal Process.*, vol. 58, pp. 6312–6323, Dec. 2010.

[17] H. V. Poor, *An Introduction to Signal Detection and Estimation*, 2nd edition. Springer, 1994.

[18] B. Nosrat-Makouei, J. Andrews, and R. W. Heath, "MIMO interference alignment over correlated channels with imperfect CSI," *IEEE Trans. Signal Process.*, vol. 59, pp. 2783–2794, June 2011.

[19] J. Gurland, "Distribution of definite and of indefinite quadratic forms," *Ann. Math. Stat.*, vol. 26, pp. 122–127, Mar. 1955.

[20] S. Kotz, N. L. Johnson, and D. W. Boyd, "Series representation of distributions of quadratic forms in normal variables—I: central case," *Ann. Math. Statist.*, pp. 823–837, June 1967.

[21] S. Kotz, N. L. Johnson, and D. W. Boyd, "Series representation of distributions of quadratic forms in normal variables—II: non-central case," *Ann. Math. Statist.*, pp. 838–848, June 1967.

[22] D. Raphaeli, "Distribution of noncentral quadratic forms in complex normal variables," *IEEE Trans. Inf. Theory*, vol. 42, pp. 1002–1007, May 1996.

[23] R. Nabar, H. Bolcskei, and A. Paulraj, "Diversity and outage performance of space-time block coded Ricean MIMO channels," *IEEE Trans. Wireless Commun.*, vol. 4, pp. 2519–2532, Sep. 2005.

[24] M. O. Hasna, M. S. Alouini, and M. K. Simon, "Effect of fading correlation on the outage probability of cellular mobile radio systems," in *Proc. 2001 IEEE Veh. Tech. Conf.*, pp. 1794–1998.

[25] T. Al-Naffouri and B. Hassibi, "On the distribution of indefinite quadratic forms in Gaussian random variables," in *Proc. 2009 IEEE Int. Symp. Inf. Theory*.

[26] A. Mathai and S. Provost, *Quadratic Forms in Random Variables: Theory and Applications*. M. Dekker, 1992.

[27] J. Park, Y. Sung, D. Kim, and H. V. Poor, "A study on the series expansion of Gaussian quadratic forms," Technical Report, Wirless Information Systems Research Lab., Dept. of EE, KAIST. http://wisrl.kaist.ac.kr/papers/techreport_wisrl_2012_apr_01.pdf, Apr. 2012.

[28] H. Sung, S. Park, K. Lee, and I. Lee, "Linear precoder designs for K-user interference channels," *IEEE Trans. Wireless Commun.*, vol. 9, pp. 291–301, Jan. 2010.

[29] I. Csiszár and P. C. Shields, *Information Theory and Statistics: A Tutorial*. Now Publishers Inc., 2004.

[30] E.-V. Belmega, S. Lasaulce, and M. Debbah, "Power allocation games for MIMO multiple access channels with coordination," *IEEE Trans. Wireless Commun.*, vol. 8, pp. 3182–3192, June 2009.

[31] A. Hjørungnes and D. Gesbert, "Precoding of orthogonal space-time block codes in arbitrarily correlated MIMO channels: iterative and closed-form solutions," *IEEE Trans. Wireless. Commun.*, vol. 6, pp. 1072–1082, Mar. 2007.

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