Information, Energy and Density for *Ad Hoc* Sensor Networks over Correlated Random Fields: Large Deviations Analysis

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***Abstract*— Using large deviations results that characterize the amount of information per node on a two-dimensional (2-D) lattice, asymptotic behavior of a sensor network deployed over a correlated random field for statistical inference is investigated. Under a 2-D hidden Gauss-Markov random field model with symmetric first order conditional autoregression, the behavior of the total information [nats] and energy efficiency [nats/J] defined as the ratio of total gathered information to the required energy is obtained as the coverage area, node density and energy vary.**

I. Introduction

In this paper, we investigate the fundamental behavior of a flat multi-hop *ad hoc* sensor network deployed over a correlated two-dimensional (2-D) random field for statistical inference. In particular, we examine the amount of information obtainable from a sensor network distributed over a 2-D Gauss-Markov random field (GMRF) and related trade-offs in various asymptotic settings. We consider the Kullback-Leibler information (KLI) and mutual information (MI) [1] as our information measures. Our approach to calculating the total obtainable information is based on the large deviations principle. That is, for large networks the total information is approximately given by the product of the number of sensors and the asymptotic per-sensor information. However, a closed-form expression for the asymptotic per-sensor information (or asymptotic information rate in 2-D) is not available for general 2-D signals. To address this problem, we adopt the *conditional autoregression (CAR) model* and corresponding correlation model for the signal, and derive a closed-form expression for the asymptotic information rate in 2-D. We do so by exploiting the spectral structure of the CAR signal and the relationship between the eigenvalues of the block circulant approximation to a block Toeplitz matrix describing the 2-D correlation structure. Based on the derived expressions for asymptotic information rate and their properties, we investigate the behavior of sensor networks deployed over correlated random fields for statistical inference.

A. Related Work

Large deviations analysis of Gauss-Markov processes in Gaussian noise has been considered previously. (See [2] and references therein.) However, most work in this area considers only one-dimensional (1-D) signals or time series. A closed-form expression for the asymptotic KLI rate was obtained and its properties were investigated for 1-D hidden Gauss-Markov random processes [2]. Large deviations analyses were used to examine the issues of optimal sensor density and optimal sampling in a 1-D signal model in [3] and [4]. For a 2-D setting, an error exponent was obtained for the detection of 2-D GMRFs in [5], where the sensors are located randomly and the Markov graph is based on the nearest neighbor dependency enabling a loop-free graph. In this work, however, measurement noise was not considered. Our work here focuses on the analysis of the fundamental behavior of 2-D sensor networks deployed for statistical inference via new large deviations results for 2-D *hidden* GMRFs, which enable us to investigate the impact of field correlation and measurement signal-to-noise ratio (SNR) on the information.

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II. Background and Signal Model

To simplify the problem and gain insights into behavior in 2-D, we assume that sensors are located on a 2-D lattice $\mathcal{I}_n = [0:1:n-1]^2$, as shown in Fig. 1. We assume that the signal samples of sensors form a (discrete-index) 2-D GMRF and that each sensor has Gaussian measurement noise. The (noisy) measurement Y_{ij} of Sensor ij on the 2-D lattice $\mathcal{I}_n$ is given by

$$Y_{ij} = X_{ij} + W_{ij}, \quad ij \in \mathcal{I}_n, \tag{1}$$

where $\{W_{ij}\}$ represents independent and identically distributed (i.i.d.) $\mathcal{N}(0, \sigma^2)$ noise with a known variance σ^2, and $\{X_{ij}\}$ is a GMRF on the 2-D lattice independent of the measurement noise $\{W_{ij}\}$. Thus, the observation samples form a 2-D hidden GMRF. In the following, we briefly introduce the results on GMRFs relevant to further development.

Definition 1 (GMRF [6]): A random vector $\mathbf{X} = (X_1, X_2, \cdots, X_n) \in \mathbb{R}^n$ is a Gauss-Markov random field with respect to (w.r.t.) a labelled graph $\mathcal{G} = (\nu, \mathcal{E})$ with mean $\boldsymbol{\mu}$ and precision matrix $\mathbf{Q} > 0$, if its probability density function is given by

$$p(\mathbf{X}) = (2\pi)^{-n/2} |\mathbf{Q}|^{1/2} \exp\left(-\frac{1}{2}(\mathbf{X}-\boldsymbol{\mu})^T \mathbf{Q} (\mathbf{X}-\boldsymbol{\mu})\right), \tag{2}$$

and $Q_{lm} \neq 0 \iff \{l, m\} \in \mathcal{E}$ for all $l \neq m$. Here, ν is the set of all nodes $\{1, 2, \cdots, n\}$ and $\mathcal{E}$ is the set of edges connecting pairs of nodes, which represent the conditional dependence structure.

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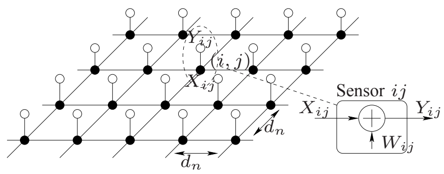

Fig. 1. Sensors on a 2-D Lattice $\mathcal{I}_n$: Hidden Markov Structure

The 2-D indexing scheme ij in (1) can be appropriately converted to an 1-D scheme to apply Definition 1. From here on, we use the 2-D indexing scheme for convenience.

Definition 2 (Stationarity): A GMRF $\{X_{ij}\}$ on a 2-D doubly infinite lattice $\mathcal{I}_\infty$ is said to be stationary if the mean vector is constant and $\mathrm{Cov}(X_{ij}, X_{i'j'}) \triangleq \mathbb{E}\{(X_{ij} - \mathbb{E}\{X_{ij}\})(X_{i'j'} - \mathbb{E}\{X_{i'j'}\})\} = c(i-i', j-j')$ for some function $c(\cdot,\cdot)$.

For a 2-D stationary GMRF $\{X_{ij}\}$, the covariance $\{\gamma_{ij}\}$ is defined as $\gamma_{ij} = \mathbb{E}\{X_{i'j'}X_{i'+i,j'+j}\} = \mathbb{E}\{X_{00}X_{ij}\}$, which does not depend on i' or j' due to the stationarity. The spectral density function of a zero-mean and stationary GMRF on $\mathcal{I}_\infty$ with covariance γ_{ij} is defined as

$$f(\omega_1,\omega_2) = \frac{1}{4\pi^2}\sum_{ij\in\mathcal{I}_\infty}\gamma_{ij}\exp(-\iota(i\omega_1 + j\omega_2)), \quad (3)$$

where $\iota = \sqrt{-1}$ and $(\omega_1,\omega_2) \in (-\pi,\pi]^2$. Note that this is a 2-D extension of the conventional 1-D discrete-time Fourier transform (DTFT).

Definition 3 (The Conditional Autoregression (CAR)): A GMRF $\{X_{ij}\}$ is said to be a conditional autoregression if it is specified using a set of full conditional normal distributions with mean and precision:

$$\mathbb{E}\{X_{ij}|\mathbf{X}_{-ij}\} = -\frac{1}{\theta_{00}}\sum_{i'j'\in\mathcal{I}_\infty\neq 00}\theta_{i'j'}X_{i+i',j+j'}, \quad (4)$$

$$\mathrm{Prec}\{X_{ij}|\mathbf{X}_{-ij}\} = \theta_{00} > 0, \quad (5)$$

where $\mathbf{X}_{-ij}$ denotes the set of all variables except X_{ij}. It is shown that the GMRF defined by the CAR model (4) - (5) is a zero-mean stationary Gaussian process on $\mathcal{I}_\infty$ with the power spectral density [6]

$$f(\omega_1,\omega_2) = \frac{1}{4\pi^2}\frac{1}{\sum_{ij\in\mathcal{I}_\infty}\theta_{ij}\exp(-\iota(i\omega_1 + j\omega_2))} \quad (6)$$

$$\text{if} \quad |\{\theta_{ij}\neq 0\}| < \infty, \quad \theta_{ij} = \theta_{-i,-j}, \quad \theta_{00} > 0, \quad (7)$$

$$\{\theta_{ij}\} \text{ is so that } f(\omega_1,\omega_2) > 0, \quad \forall(\omega_1,\omega_2)\in(-\pi,\pi]^2. \quad (8)$$

Henceforth, we assume that the 2-D stochastic signal $\{X_{ij}\}$ in (1) is given by a stationary GMRF defined by the CAR model (4) - (5) and (7) - (8).

III. Asymptotic Information Rates and Their Properties

In this section, we derive a closed-form expression for the asymptotic KLI rate and MI rate in the model (1), defined as

$$\mathcal{K} = \lim_{n\to\infty}\frac{1}{|\mathcal{I}_n|}\log\frac{p_0}{p_1}(\{Y_{ij}, ij\in\mathcal{I}_n\}) \text{ a.s. under } p_0, \quad \text{and}$$

$$I = \lim_{n\to\infty}\frac{1}{|\mathcal{I}_n|}I(\{X_{ij}, ij\in\mathcal{I}_n\};\{Y_{ij}, ij\in\mathcal{I}_n\}),$$

respectively. For the MI, the signal model (1) is directly applicable, whereas for the KLI the probability density functions of the null (noise-only) and alternative (signal-plus-noise) distributions are given by

$$p_0(Y_{ij}) \;:\; Y_{ij} = W_{ij}, \quad ij\in\mathcal{I}_n,$$

$$p_1(Y_{ij}) \;:\; Y_{ij} = X_{ij} + W_{ij}, \quad ij\in\mathcal{I}_n. \quad (9)$$

The following closed-form expressions for the asymptotic information rates in the spectral domain have been obtained in [7] by exploiting the spectral structure of the CAR signal and the relationship between the eigenvalues of block circulant and block Toeplitz matrices representing 2-D correlation structure.

Theorem 1: For the model (9) with the signal given by (4) - (5), assuming that conditions (7) - (8) hold, the asymptotic KLI rate is given by

$$\begin{aligned}\mathcal{K} &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}\log\frac{\sigma^2 + 4\pi^2 f(\omega_1,\omega_2)}{\sigma^2} \right. \quad (10)\\ &\qquad \left. + \frac{1}{2}\frac{\sigma^2}{\sigma^2 + 4\pi^2 f(\omega_1,\omega_2)} - \frac{1}{2}\right)d\omega_1 d\omega_2,\\ &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}D(\mathcal{N}(0,S_0^y(\omega_1,\omega_2))||\mathcal{N}(0,S_1^y(\omega_1,\omega_2))\; d\omega_1 d\omega_2,\end{aligned}$$

where $D(\cdot||\cdot)$ denotes the Kullback-Leibler divergence.
Proof: In [8].

As a by-product of the proof of the above theorem, we have the asymptotic MI rate given by

$$I = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1}{2}\log\frac{\sigma^2 + 4\pi^2 f(\omega_1,\omega_2)}{\sigma^2}d\omega_1 d\omega_2. \quad (11)$$

Theorem 1 is a 2-D extension of the asymptotic KLI rate of 1-D hidden Gauss-Markov model obtained in [2], and the asymptotic KLI rate (10) can be explained using a frequency binning argument. Specifically, for each 2-D frequency bin $d\omega_1 d\omega_2$, the spectra are flat, i.e., the signals are independent and Stein's lemma can be applied for the bin. The overall KLI is the sum of contributions from each segment.

A. Symmetric First Order Conditional Autoregression

To investigate the properties of the asymptotic KLI and MI rates as functions of field correlation and SNR, we further consider the symmetric first order conditional autoregression (SFCAR), defined by the conditions

$$\mathbb{E}\{X_{ij}|\mathbf{X}_{-ij}\} = \frac{\lambda}{\kappa}(X_{i+1,j} + X_{i-1,j} + X_{i,j+1} + X_{i,j-1}),$$

$$\mathrm{Prec}\{X_{ij}|\mathbf{X}_{-ij}\} = \kappa > 0,$$

where $0 \le \lambda \le \frac{\kappa}{4}$. (This is a sufficient condition to satisfy (7) - (8).) Here, $\theta_{00} = \kappa$ and $\theta_{1,0} = \theta_{-1,0} = \theta_{0,1} = \theta_{0,-1} = -\lambda$. In the SFCAR model, the correlation is symmetric for each set of four neighboring sensor nodes. The SFCAR model is a simple but meaningful extension of the 1-D autoregression (AR) model which has the conditional causal dependency only on the previous sample. Here in the 2-D case we have conditional dependence on four neighboring nodes in the four (planar) directions, capturing 2-D correlation structure. The spectrum of the SFCAR signal is given by

$$f(\omega_1,\omega_2) = \frac{1}{4\pi^2\kappa(1 - 2\zeta\cos\omega_1 - 2\zeta\cos\omega_2)}, \quad (12)$$

where the *edge dependence factor* ζ is defined as

$$\zeta \triangleq \frac{\lambda}{\kappa}, \quad 0 \leq \zeta \leq 1/4. \tag{13}$$

Here, $\zeta = 0$ corresponds to the i.i.d. case whereas $\zeta = 1/4$ corresponds to the perfectly correlated case. Therefore, the correlation strength can be captured in this single quantity ζ for SFCAR signals. The power of the SFCAR is obtained using the inverse Fourier transform via the relationship (3), and is given by $P_s = \gamma_{00} = \frac{2K(4\zeta)}{\pi\kappa}$, $\left(0 \leq \zeta \leq \frac{1}{4}\right)$, where $K(\cdot)$ is the complete elliptic integral of the first kind [9]. The SNR is given by $\text{SNR} = \frac{P_s}{\sigma^2} = \frac{2K(4\zeta)}{\pi\kappa\sigma^2}$. Using (10) and the SNR, we obtain the asymptotic KLI and MI rates for the SFCAR signal, given in the following corollary to Theorem 1, also from [7].

Corollary 1: The asymptotic KLI and MI rates for the SFCAR 2D signal model are given by

$$\begin{aligned}\mathcal{K}_s &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Bigg(\frac{1}{2}\log\left(1+\frac{\text{SNR}}{(2/\pi)K(4\zeta)(1-2\zeta\cos\omega_1-2\zeta\cos\omega_2)}\right) \\ &\quad + \frac{1}{2}\frac{1}{1+\frac{\text{SNR}}{(2/\pi)K(4\zeta)(1-2\zeta\cos\omega_1-2\zeta\cos\omega_2)}} - \frac{1}{2}\Bigg)d\omega_1 d\omega_2. \end{aligned} \tag{14}$$

and

$$I_s = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1}{2}\log\left(1+\frac{\text{SNR}}{(2/\pi)K(4\zeta)(1-2\zeta\cos\omega_1-2\zeta\cos\omega_2)}\right)d\omega_1 d\omega_2, \tag{15}$$

respectively.

Note that the SNR and correlation are separated in (14)-(15), which enables us to investigate the effects of each term separately.

B. Properties of the asymptotic KLI and MI rates ($\mathcal{K}_s$ and I_s)

First, it is readily seen from Corollary 1 that $\mathcal{K}_s$ and I_s are continuously differentiable C^1 functions of the edge dependence factor ζ $(0 \leq \zeta \leq 1/4)$ for a given SNR since $f : x \to K(x)$ is a continuously differentiable C^∞ function for $0 \leq x < 1$ [10]. The values of $\mathcal{K}_s$ at the extreme correlations are given by noting that $K(0) = \frac{\pi}{2}$ and $K(1) = \infty$. Therefore, in the i.i.d. case ($\zeta = 0$), the corollary reduces to Stein's lemma as expected, and $\mathcal{K}_s$ is given by

$$\mathcal{K}_s|_{\zeta=0} = \frac{1}{2}\log(1+\text{SNR}) + \frac{1}{2(1+\text{SNR})} - \frac{1}{2} = D(\mathcal{N}(0,1)||\mathcal{N}(0,1+\text{SNR})).$$

In the i.i.d. case, the asymptotic MI rate is given by the well known formula, $I_s|_{\zeta=0} = \frac{1}{2}\log(1+\text{SNR})$. For the perfectly correlated case ($\zeta = 1/4$), on the other hand, $\mathcal{K}_s = 0$ and $I_s = 0$. (In this case as well as in the i.i.d. case, the two-dimensionality is irrelevant.) The limiting behavior of the asymptotic information rates is given by Taylor's theorem. Due to the continuous differentiability, we have

$$\mathcal{K}_s(\zeta) = c_1 \cdot (1/4 - \zeta) + o(|1/4 - \zeta|), \tag{16}$$

$$I_s(\zeta) = c_1' \cdot (1/4 - \zeta) + o(|1/4 - \zeta|), \tag{17}$$

for some constants c_1 and c_1', as $\zeta \to 1/4$. Similarly, we also have the linear limiting behavior for $\mathcal{K}_s$ and I_s in a neighborhood of $\zeta = 0$ with non-zero limit values, as $\zeta \to 0$. That is,

$$\mathcal{K}_s(\zeta) = \mathcal{K}_s(0) + c_2\zeta + o(\zeta), \tag{18}$$

$$I_s(\zeta) = I_s(0) + c_2'\zeta + o(\zeta), \tag{19}$$

for some c_2 and c_2', as $\zeta \to 0$. For intermediate values of correlation, it is seen that at high SNR $\mathcal{K}_s$ is monotonically decreasing as ζ increases. At low SNR, on the other hand, correlation is beneficial to the performance.

With regard to $\mathcal{K}_s$ and I_s as functions of SNR, the behavior of $\mathcal{K}_s$ is given by the following theorem.

Theorem 2: The asymptotic KLI rate $\mathcal{K}_s$ for the hidden SFCAR model is continuous and monotonically increasing as SNR increases for a given edge dependence factor $0 \leq \zeta < 1/4$. Moreover, $\mathcal{K}_s$ increases linearly with respect to $\frac{1}{2}\log\text{SNR}$ as $\text{SNR} \to \infty$. As SNR decreases to zero, on the other hand, $\mathcal{K}_s$ converges to zero with the convergence rate $\mathcal{K}_s(\text{SNR}) = c_3 \cdot \text{SNR}^2 + o(\text{SNR}^2)$ for some constant c_3 as $\text{SNR} \to 0$. The asymptotic MI rate I_s has similar properties as a function of SNR, i.e., it is a continuous and monotonically-increasing function of SNR. At high SNR, it increases with rate $\frac{1}{2}\log\text{SNR}$, whereas it decreases to zero with rate of convergence $I_s(\text{SNR}) = c_3' \cdot \text{SNR} + o(\text{SNR})$ for some constant c_3' as $\text{SNR} \to 0$.

Proof: In [8].

Note that the limiting behavior as $\text{SNR} \to 0$ is different for $\mathcal{K}_s$ and I_s; $\mathcal{K}_s$ decays to zero quadratically while I_s diminishes linearly.

IV. Scaling Laws in Ad Hoc Sensor Networks over Correlated Random Field

Based on the results in the previous sections, we are now ready to answer some fundamental questions in the design of sensor networks for statistical inference about the underlying stochastic field.

A. Physical correlation model

The actual physical correlation for the SFCAR model is given by solving the corresponding continuous-index 2-D stochastic differential equation (the stochastic Laplace equation)[1] [11]

$$\left[\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 - \alpha^2\right]X(x,y) = u(x,y), \tag{20}$$

where $u(x,y)$ is the 2-D white zero-mean Gaussian perturbation and $\alpha > 0$ is the diffusion rate. By solving the SDE, the *edge correlation factor* ρ is given, as a function of the sensor spacing d_n, by [11]

$$\rho \triangleq \frac{\gamma_{01}}{\gamma_{00}} = \frac{\gamma_{10}}{\gamma_{00}} = f(d_n) = \alpha d_n K_1(\alpha d_n), \tag{21}$$

where $K_1(\cdot)$ is the modified Bessel function of the second kind whose asymptotic behavior is given by

$$\begin{cases} K_1(x) \to \sqrt{\frac{\pi}{2x}}e^{-x} & \text{as } x \to \infty, \\ K_1(x) \to 1/x & \text{as } x \to 0. \end{cases} \tag{22}$$

The correlation function (21) can be regarded as the representative correlation in 2-D, similar to the exponential correlation function e^{-Ad_n} in 1-D. Both functions decrease monotonically w.r.t. d_n. However, the 2-D correlation function is flat at

[1] Note that the solution of (20) is circularly symmetric, i.e., it depends only on $r = \sqrt{x^2+y^2}$, and samples of the solution $X(x,y)$ of (20) on lattice $\mathcal{I}_n$ do not necessarily form a discrete-index SFCAR GMRF. However, (20) is still the continuous-index counterpart of the SFCAR model, and we use its correlation function for the SFCAR model.

$d_n = 0$ [11]. Further, we have a continuous and differentiable mapping $g : \rho \to \zeta$ from the edge correlation factor ρ to the edge dependence factor ζ, given by [8]

$$\rho = \frac{(2/\pi)K(4\zeta) - 1}{4(2/\pi)\zeta K(4\zeta)} =: g^{-1}(\zeta), \tag{23}$$

which maps zero and one to zero and 1/4, respectively. Thus, we have $\zeta = g(f(d_n))$, and for given physical parameters (with a slight abuse of notation),

$$\mathcal{K}_s(\text{SNR}, \zeta) = \mathcal{K}_s(\text{SNR}, g(f(d_n))) = \mathcal{K}_s(\text{SNR}, d_n).$$

(And, similarly for I_s.) We will use the arguments SNR and ζ for $\mathcal{K}_s$ and I_s properly if necessary.

B. Asymptotic behavior

In the following, we summarize the assumptions for the planar *ad hoc* sensor network that we consider.

(A.1) n^2 sensors are located on the grid $\mathcal{I}_n = [0 : 1 : n-1]^2$ with spacing d_n, as shown in Fig. 1, and a fusion center is located at the center $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$.

(A.2) The observations $\{Y_{ij}\}$ at sensor nodes form a 2-D hidden (discrete-index) SFCAR Gauss-Markov random field on the lattice for each $d_n > 0$, and the edge dependence factor is given by (21) and (23).

(A.3) The fusion center gathers the measurement from all nodes using the minimum hop routing. Note that the links in Fig. 1 are not only the Markov dependence edges but also the routing links. The minimum hop routing requires a hop count of $|i - \lfloor n/2 \rfloor| + |j - \lfloor n/2 \rfloor|$ to deliver Y_{ij} to the fusion center.

(A.4) The communication energy per link $E_c(d_n) = E_0 d_n^{\nu}$, where $\nu \geq 2$ is the propagation loss factor in wireless channel.

(A.5) Sensing requires energy, and the sensing energy per node is denoted by E_s. Moreover, we assume that the *measurement* SNR increases linearly w.r.t. E_s, i.e., $\text{SNR} = \beta E_s$ for some constant β.

Henceforth, we consider various asymptotic scenarios and investigate the fundamental behavior of the *ad hoc* sensor network deployed over a correlated random field for statistical inference under assumptions *(A.1)-(A.5)*. (Proofs are omitted due to limited space.)

The sensor density μ_n on $\mathcal{I}_n$ is given by $\mu_n = \frac{n^2}{((n-1)d_n)^2}$. Assuming that the network is sufficiently large, the total information about the underlying field obtainable from the network is given by

$$\text{KLI}_T = n^2 \mathcal{K}_s \text{ and } \text{MI}_T = n^2 I_s, \tag{24}$$

and the total consumed energy in the network is given by

$$\begin{aligned} E &= n^2 E_s + E_c(d_n) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (|i - \lfloor n/2 \rfloor| + |j - \lfloor n/2 \rfloor|), \\ &= n^2 E_s + \Theta(n^3) E_c(d_n). \end{aligned} \tag{25}$$

Note that the knowledge of per-node information $\mathcal{K}_s$ and I_s and their properties w.r.t. SNR and sensor spacing d_n in (24) is critical for further development, and it is provided in the previous sections.

We begin with the increasing area case.

Theorem 3 (Infinite area and fixed density): For an *ad hoc* sensor network with a fixed and finite node density, the total amount of information increases linearly as the area increases, but under both information measures the amount of harvested information per unit energy decays to zero with rate

$$\eta = \Theta\left(\text{area}^{-1/2}\right), \tag{26}$$

for any non-trivial diffusion rate α, i.e., $0 < \alpha < \infty$ as we increase the area.

Next, we consider the case in which the node density diminishes, i.e., $d_n \to \infty$. This case is of particular interest at high SNR since at high SNR less correlated samples yield larger per-node information. However, the per-sensor information is upper bounded as $d_n \to \infty$, and the asymptotic behavior is given by the following theorem.

Theorem 4: As $d_n \to \infty$, the per-node information $\mathcal{K}_s$ and I_s converge to $\mathcal{K}_s(0) = D(\mathcal{N}(0,1)||\mathcal{N}(0, 1+\text{SNR}))$ and $I_s(0) = \frac{1}{2}\log \text{SNR}$, respectively, and the convergence rate is given by

$$\mathcal{K}_s(d_n) = \mathcal{K}_s(0) - c_4\sqrt{d_n}e^{-\alpha d_n} + o\left(\sqrt{d_n}e^{-\alpha d_n}\right), \tag{27}$$

$$I_s(d_n) = I_s(0) - c_4'\sqrt{d_n}e^{-\alpha d_n} + o\left(\sqrt{d_n}e^{-\alpha d_n}\right), \tag{28}$$

for constants $c_4,\ c_4' > 0$ depending on the SNR.

Theorem 4 can be proved using (18, 19) and (21, 22), and explains how much gain is obtained from less correlated observations by increasing the sensor spacing in 2-D. Fig. 2 shows $\mathcal{K}_s$ and E_c as functions of d_n for $\alpha = 1$, $c_4 = 1$ and 10 dB SNR. The gain in information is given by $\sqrt{d_n}e^{-\alpha d_n}$ for large d_n, whereas the required per-link communication energy increases without bound, i.e., $E_c(d_n) = E_0 d_n^{\nu}$ $(\nu \geq 2)$. Since the exponential term is dominant in the gain as d_n increases, the information gain obtained by increasing d_n decreases almost exponentially, and there is no significant gain by increasing the sensor spacing further after some value. Hence, it is not effective in terms of energy efficiency to deploy a very sparse network aiming at less correlated samples at high SNR.

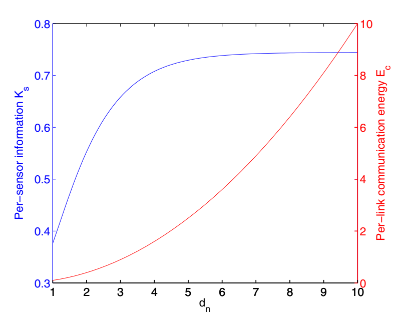

Fig. 2. Per-node information and per-link communication energy w.r.t. sensor spacing d_n (SNR = 10 dB, $\alpha = 1$, $c_4 = 1$)

The per-link communication energy can be made arbitrarily small by decreasing the sensor spacing. To investigate the effect of diminishing communication energy E_c as $d_n \to 0$, we now consider the asymptotic case in which the node density goes to infinity for a fixed coverage area. In this case, the per-node information decays to zero as $d_n \to 0$ since $\zeta \to 1/4$ as $d_n \to 0$, and $\mathcal{K}_s(\zeta)$ and $I_s(\zeta)$ converge to zero as $\zeta \to 1/4$, as shown in Section III-B. The asymptotic behavior in this case is given by the following theorem.

Theorem 5 (Infinite density model): For the infinite density model with a fixed coverage area, the per-node information decays to zero with rate

$$\mathcal{K}_s = c_5 \mu_n^{-1} + o\left(\mu_n^{-1}\right), \tag{29}$$

for some constant c_5 as the node density $\mu_n \to \infty$. Hence, the amount of total information per unit area (nats/m^2) converges to the constant c_5 as $\mu_n \to \infty$. Furthermore, in the case of no sensing energy, a non-zero energy efficiency η is achievable if the propagation loss factor $\nu = 3$, and even an infinite energy efficiency is achievable if $\nu > 3$ as $\mu_n \to \infty$ for fixed area.[2]

The finite total information for the infinite density and fixed area model follows our intuition. The maximum information provided by the samples from the continuous-index random field does not exceed the information between $X(x, y)$ and $Y(x, y)$ except for the case of spatially white fields. It is common that the propagation loss factor $\nu > 3$ for near field propagation (i.e., $d_n \to 0$). Hence, infinite energy efficiency is achievable as we increases the node density for a fixed area considering only communication energy. Note that the total amount of information converges to a constant as we increases the node density. So, the infinite energy efficiency is achieved by diminishing communication energy as $d_n \to 0$. Considering the sensing energy, however, infinite energy efficiency is not feasible since we have in this case

$$E = n^2 E_s + \Theta(n^{3-\nu}) \text{ and } \eta = \frac{c_5 + o(1)}{n^2 E_s + \Theta(n^{3-\nu})}, \quad \nu \geq 2, \tag{30}$$

as $n \to \infty$ for fixed coverage area. In this case the sensing energy $n^2 E_s$ is the dominant factor for low energy efficiency, and the energy efficiency decreases to zero with rate $O\left(\mu_n^{-1}\right)$. Thus, it is critical for a densely deployed sensor network to minimize the sensing energy or processing energy for each sensor.

In the infinite density model, we have observed that energy is an important factor in efficiency. Now we investigate the change of total information w.r.t. energy. We fix the node density and consider two scenarios to increase the required energy: One is to fix the coverage area also and increase the sensing energy, and the other is to fix the sensing energy and increase the coverage area. We assume that the network size is sufficiently large so that our asymptotic analysis is valid. The energy asymptotic behavior for two scenarios is summarized in the following theorem.

[2]Of course, this depends on the assumption of $E_c(d_n) = E_0 d_n^{\nu}$ for any $d_n > 0$. However, this assumption may not be valid for small d_n.

Theorem 6: As we increase the total energy E consumed by a sensor network with a fixed node density and fixed area, the total information increases with rate

$$\text{Total information} = O\left(\log E\right) \tag{31}$$

as $E \to \infty$. When the node density and sensing energy are fixed and the increasing energy is used to enlarge the coverage area, on the other hand, the total amount of information increases with rate of

$$\text{Total information} = \Theta\left(E^{2/3}\right), \tag{32}$$

for any $\nu > 0$, as $E \to \infty$.

Theorem 6 suggests a guideline for investing the excess energy. It is not efficient to invest energy to improve the quality of sensed samples from a limited area. This only provides the increase in total information in logarithmic scale. Rather the energy should be spent to increase the number of samples by enlarging the coverage area even if it yields less accurate samples.

V. Conclusions

We have analyzed the asymptotic behavior of ad hoc sensor networks deployed over correlated random field for statistical inference. Using our large deviations results that characterize the asymptotic information rate in 2-D for GMRFs under the CAR model, we have obtained fundamental scaling laws for total information and energy efficiency as the node density, coverage area and consumed energy change. The results provide guidelines for sensor network design for statistical inference about 2-D correlated random fields such as temperature, humidity, density of a gas on a certain area.

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