Simpler Condition for Theorem 2

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*T*heorem 2 (Asymptotic KLI rate in d-D] Suppose that
A.1 the alternative spectrum $f_1(\boldsymbol{\omega})$ has a positive lower bound, and
A.2 $\exists\, M < \infty$ such that $\forall\, k = 1, 2, \cdots, d, \quad \sum_{\mathbf{h} \in \mathbb{Z}^d} (1 + |h_k|)|\gamma_{\mathbf{h}}| < M$.
Then, the asymptotic KLI rate $\mathcal{K}$ for (24) is given by

$$\mathcal{K} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi)^d} \left[\frac{1}{2} \log \frac{(2\pi)^d f_1(\boldsymbol{\omega})}{\sigma^2} - \frac{1}{2} \left(1 - \frac{\sigma^2}{(2\pi)^d f_1(\boldsymbol{\omega})} \right) \right] d\boldsymbol{\omega}, \tag{1}$$

$$= \frac{1}{(2\pi)^d} \int_{[-\pi,\pi)^d} D(\mathcal{N}(0, \sigma^2) || \mathcal{N}(0, (2\pi)^d f_1(\boldsymbol{\omega}))) d\boldsymbol{\omega}, \tag{2}$$

where $D(\cdot||\cdot)$ denotes the Kullback-Leibler distance.

Lemma 1: (Guyon, 1995 [27, Page 2]) If $\mathbf{k} = (k_1, \cdots, k_d) \in \mathbb{N}^d$ and if $f_1(\boldsymbol{\omega})$ is of class $C^{\mathbf{k}}$ (i.e., differentiable up to the k_d-order w.r.t. ω_d), then

$$\limsup_{\mathbf{h} \to \infty} h_1^{k_1} h_2^{k_2} \cdots h_d^{k_d} |\gamma_{\mathbf{h}}| < \infty, \tag{3}$$

where $\mathbb{N}$ is the set of all natural numbers, and $\mathbf{h} \to \infty$ means that at least one coordinate tends to infinity.

Using the above lemma, (A.2) of Theorem 2 can be modified as an intuitive one, given in Theorem 2'.

*T*heorem 2' (Asymptotic KLI rate in d-D) Suppose that
A.1 the alternative spectrum $f_1(\boldsymbol{\omega})$ has a positive lower bound, and
A.2 $f_1(\boldsymbol{\omega})$ is more than twice differentiable.
Then, the asymptotic KLI rate $\mathcal{K}$ for (24) is given by

$$\mathcal{K} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi)^d} \left[\frac{1}{2} \log \frac{(2\pi)^d f_1(\boldsymbol{\omega})}{\sigma^2} - \frac{1}{2} \left(1 - \frac{\sigma^2}{(2\pi)^d f_1(\boldsymbol{\omega})} \right) \right] d\boldsymbol{\omega}, \tag{4}$$

$$= \frac{1}{(2\pi)^d} \int_{[-\pi,\pi)^d} D(\mathcal{N}(0, \sigma^2) || \mathcal{N}(0, (2\pi)^d f_1(\boldsymbol{\omega}))) d\boldsymbol{\omega}, \tag{5}$$

where $D(\cdot||\cdot)$ denotes the Kullback-Leibler distance.

Proof: The proof is by showing that more than twice differentiability of spectrum is sufficient to satisfy the assumptions of Theorem 2. By Lemma 1, we have

$$|\gamma_{h_1,h_2,\cdots,h_d}| = O\left(\frac{1}{h_1^{2+\epsilon} h_2^{2+\epsilon} \cdots h_d^{2+\epsilon}} \right).$$

since $f_1(\omega_1, \omega_2, \cdots, \omega_d) \in C^{(2+\epsilon, 2+\epsilon, \cdots, 2+\epsilon)}$. Then,

$$\sum_{(h_1,h_2,\cdots,h_d) \in \mathbb{Z}^d} |h_1||\gamma_{h_1,h_2,\cdots,h_d}| = \sum_{h_1} O(|h_1|^{-1-\epsilon}) \sum_{h_2} O\left(\frac{1}{h_2^{2+\epsilon}}\right) \cdots \sum_{h_d} O\left(\frac{1}{h_d^{2+\epsilon}}\right),$$

$$< \infty.$$

Similarly, for $2 \le k \le d$, we have $\sum_{(h_1,h_2,\cdots,h_d) \in \mathbb{Z}^d} |h_k||\gamma_{h_1,h_2,\cdots,h_d}| < \infty$. Hence, more than twice differentiability of spectrum is sufficient for (A.2) and the strong convergence. ■

Weaker conditions may be possible. Note that in 1-D case, a known sufficient condition for strong convergence is that the null and alternative spectrum have finite lower and upper bound, and are continuous and strictly positive. Also, the absolute summability of the covariance function guarantees a well defined spectrum (i.e., upper bounded spectrum) in 1-D case.