

Optimal and Suboptimal Detection of Gaussian Signals in Noise: Asymptotic Relative Efficiency

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ABSTRACT

The performance of Bayesian detection of Gaussian signals using noisy observations is investigated via the error exponent for the average error probability. Under unknown signal correlation structure or limited processing capability it is reasonable to use the simple quadratic detector that is optimal in the case of an independent and identically distributed (i.i.d.) signal. Using the large deviations principle, the performance of this detector (which is suboptimal for non-i.i.d. signals) is compared with that of the optimal detector for correlated signals via the asymptotic relative efficiency defined as the ratio between sample sizes of two detectors required for the same performance in the large-sample-size regime. The effects of SNR on the ARE are investigated. It is shown that the asymptotic efficiency of the simple quadratic detector relative to the optimal detector converges to one as the SNR increases without bound for any bounded spectrum, and that the simple quadratic detector performs as well as the optimal detector for a wide range of the correlation values at high SNR.

Keywords: Quadratic detector, error exponent, large deviations principle, asymptotic relative efficiency (ARE)

1. INTRODUCTION

We consider in this paper the optimal and suboptimal detection of stationary Gaussian signals using noisy observations y_i under a Bayesian formulation. The corresponding null and alternative hypotheses are given by

$$\begin{aligned} H_0 &: y_i = w_i, \quad i = 1, 2, \dots, n, \\ H_1 &: y_i = w_i + \theta s_i, \quad i = 1, 2, \dots, n, \end{aligned} \quad (1)$$

where $\{w_i\}$ is independent and identically distributed (i.i.d.) $\mathcal{N}(0, \sigma^2)$ noise with a known variance σ^2 , θ is a nonnegative constant, and $\{s_i\}$ is a zero-mean unit-variance stationary Gaussian signal with spectrum $f_s(\omega)$, independent of the noise $\{w_i\}$. The prior probabilities for the hypotheses are denoted by

$$\pi_0 \triangleq \Pr\{H_0\}, \quad \pi_1 \triangleq \Pr\{H_1\} = 1 - \pi_0. \quad (2)$$

Due to the stationarity of the signal, the signal-to-noise ratio (SNR) for the observations is constant and is given by

$$\text{SNR} = \frac{\theta^2}{\sigma^2}. \quad (3)$$

Such a model arises, for example, in sensor networks (see, e.g., Sung *et al.*^{18,19}). For a large sensor network deployed for the detection of stochastic signals such as gases or particles in a fixed area, it is reasonable to assume that the signal is random and that spatial signal samples are correlated, while the measurement noise is independent from sensor to sensor. Typically, the optimal detector for (1) is given in the form of a quadratic detector that uses the correlation structure and requires the joint processing of all signal samples. In general, optimal detection using n samples requires $O(n^2)$ multiplications and $O(n)$ memory size for storing past samples except in some cases where recursive techniques are available.⁶ These processing requirements may be prohibitive in applications such as sensor network in which each sensor node has stringent energy and storage constraints

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and the number of nodes (or observation samples) is large. Thus, one can consider other detector structures with reduced complexity, e.g., simple quadratic detectors or banded-quadratic detectors.^{7,8}

In this paper, we are interested in the asymptotic performance of these detectors and the performance comparison between them using the *asymptotic relative efficiency* (ARE) derived from the large deviations principle (LDP).⁴ Poor and Chang investigated the performance of these detectors using Pitman's ARE or asymptotic deflection ratio.^{3,7,8} While ARE from the large deviations principle is based on the law of large numbers, Pitman's ARE relies on convergence in distribution (of the test statistics). Thus, these two ARE's do not necessarily provide the same order for the performance of two detectors under consideration, and Pitman's ARE generally provides more accurate results than that of LDP in the low SNR regime.² However, Pitman's ARE is based on the asymptotic local scenario wherein the signal power decreases to zero with a certain rate, i.e., typically θ in (1) decreases as $\frac{h}{\sqrt{n}}$ for $h > 0$ as the number n of samples increases. Thus, it does not allow the performance comparison for a fixed signal-to-noise ratio (SNR). Poor and Chang considered the locally optimal detector as the reference detector under the Neyman-Pearson formulation. (The efficacy* of the optimal quadratic detector is difficult to obtain since the amplitude parameter θ is inseparable in the optimal test statistic, as shown in (14)).

The LDP for stationary Gaussian processes is well-established.¹²⁻¹⁶ Based on the result of Bryc and Dembo,¹³ here we extend the work of Poor and Chang^{7,8} and compare the relative performance of several quadratic detectors using the ARE from the LDP, focusing on the effects of SNR on the ARE with the optimal detector as the reference detector under a Bayesian formulation.

The paper is organized as follows. In Section 2, some relevant results concerning the LDP are presented. In Section 3, the quadratic detectors that we consider and the corresponding ARE are provided. In Section 4, some numerical results are presented for several examples of signal correlation, followed by the conclusion in Section 5.

2. PRELIMINARIES

In this section, we present some definitions and results concerning LDP relevant to the further development.

DEFINITION 2.1 (LARGE DEVIATIONS PRINCIPLE¹¹). *Let $\{P_n\}$ be a sequence of probability distributions defined on $(\mathcal{X}, \mathcal{F})$. $\{P_n\}$ is said to satisfy the large deviation principle with a rate function $I : \mathcal{X} \rightarrow [0, \infty]$ if*

- *the level sets $I^{-1}([0, c])$ are compact for all $c < \infty$,*

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$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log P_n(C) \leq - \inf_{x \in C} I(x) \quad \forall \text{ closed } C \in \mathcal{F},$$

- *and*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log P_n(O) \geq - \inf_{x \in O} I(x) \quad \forall \text{ open } O \in \mathcal{F}.$$

For the probability distributions governing a sequence of sample means the LDP is given by Crámer's theorem, and its extension to general sequences of random variables is provided by the Gärtner-Ellis theorem based on the convergence of cumulant generating functions.^{10,11} In particular, for the sequence of quadratic functionals of Gaussian processes the rate function is derived by Bryc and Dembo¹³ circumventing difficulties in applying the Gärtner-Ellis theorem to this problem, which is summarized in the following theorem.

THEOREM 2.2 (BRYC AND DEMBO¹³). *Let $\{Y_i, -\infty < i < \infty\}$ be a (real-valued) zero-mean stationary Gaussian process with bounded spectral density function $S_y(\omega)$ defined as*

$$S_y(\omega) = \sum_{k=-\infty}^{\infty} \mathbb{E}\{Y_0 Y_k\} e^{-jk\omega} \tag{4}$$

*Pitman's ARE is expressed by the ratio of the efficacy of one detector to that of the other.

with essential supremum M . Let a random variable $Z_n \triangleq \{\frac{1}{n} \sum_{i=1}^n Y_i^2\}$ and P_n be the distribution of Z_n , i.e., $P_n(S) \triangleq \Pr\{Z_n \in S\}$ for $S \in \mathcal{B}(\mathbb{R})$. Then, $\{P_n\}$ satisfies the LDP with a rate function

$$I(z) = \sup_{-\infty < t < \frac{1}{2M}} [zt - \Lambda(t)], \quad (5)$$

where

$$\Lambda(t) = -\frac{1}{4\pi} \int_0^{2\pi} \log(1 - 2tS_y(\omega))d\omega \quad (6)$$

LEMMA 2.3 (BRYC AND DEMBO¹³). Suppose $\mathbf{Y} = [Y_1, \dots, Y_n]^T$ is a real-valued zero-mean Gaussian vector with the covariance matrix Σ and let \mathbf{W} be a symmetric real-valued $n \times n$ matrix. Then, with $\lambda_1, \dots, \lambda_n$ the eigenvalues of the matrix $\mathbf{W}\Sigma$ we have

$$\log \mathbb{E}e^{s\mathbf{Y}^T\mathbf{W}\mathbf{Y}} = -\frac{1}{2} \sum_{i=1}^n \log(1 - 2s\lambda_i)$$

for all $s \in \mathbb{C}$ s.t. $\max_i \{Re(s)\lambda_i\} < 1/2$. Furthermore, $\log \mathbb{E}e^{t\mathbf{Y}^T\mathbf{W}\mathbf{Y}} = \infty$ for all $t \in \mathbb{R}$ s.t. $\max_i \{t\lambda_i\} \geq 1/2$.

Another useful result concerns the asymptotic distribution of the eigenvalues of a Toeplitz matrix, which is summarized in the following theorem.

THEOREM 2.4 (GRENANDER AND SZEGÖ⁹). Let $S_y(\omega)$ be the spectrum of $\{Y_i\}$, defined as (4), with finite lower and upper bounds denoted by m and M , respectively. Let $\Sigma_{y,n}$ be a covariance matrix defined as

$$\Sigma_{y,n} = [\mathbb{E}\{Y_i Y_j\}]_{i,j=1}^n \quad (7)$$

and $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ be the eigenvalues of $\Sigma_{y,n}$. Then, for any continuous function $h : [m, M] \rightarrow \mathbb{R}$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n h(\lambda_i^{(n)}) = \frac{1}{2\pi} \int_0^{2\pi} h(S_y(\omega))d\omega. \quad (8)$$

3. ASYMPTOTIC RELATIVE EFFICIENCY

In this section, we present the classes of detectors that we consider and their corresponding rate functions. By stacking the observations and corresponding signals and noises, the hypotheses (1) can be rewritten in vector form as

$$\begin{aligned} H_0 &: \mathbf{y}_n = \mathbf{w}_n, \\ H_1 &: \mathbf{y}_n = \mathbf{w}_n + \theta \mathbf{s}_n, \end{aligned} \quad (9)$$

where

$$\mathbf{y}_n \triangleq [y_0, \dots, y_n]^T, \quad \mathbf{s}_n \triangleq [s_0, \dots, s_n]^T, \quad \mathbf{w}_n \triangleq [w_0, \dots, w_n]^T,$$

and the noise vector $\mathbf{w}_n \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$, $\mathbf{s}_n \sim \mathcal{N}(\mathbf{0}, \Sigma_{s,n})$, and \mathbf{y}_n has distribution $\mathcal{N}(\mathbf{0}, \Sigma_{j,n})$ for hypothesis j ($j = 0, 1$) where

$$\Sigma_{0,n} = \sigma^2 \mathbf{I}, \quad \Sigma_{1,n} = \sigma^2 \mathbf{I} + \theta^2 \Sigma_{s,n}. \quad (10)$$

For convenience, we further assume equal prior probabilities, i.e.,

$$\pi_0 = \pi_1 = \frac{1}{2}. \quad (11)$$

Then, the optimal detector for (9) is given by the maximum *a posteriori* probability detector:

$$\delta_o(\mathbf{y}_n) = \begin{cases} 1, & \frac{1}{n} \log L_n(\mathbf{y}_n) \geq \tau = 0, \\ 0, & \text{otherwise,} \end{cases} \quad (12)$$

where

$$L_n(\mathbf{y}_n) = \left(\frac{|\boldsymbol{\Sigma}_{0,n}|}{|\boldsymbol{\Sigma}_{1,n}|} \right)^{1/2} e^{\frac{1}{2} \mathbf{y}_n^T \mathbf{Q}_n \mathbf{y}_n}, \quad (13)$$

and

$$\mathbf{Q}_n = \boldsymbol{\Sigma}_{0,n}^{-1} - \boldsymbol{\Sigma}_{1,n}^{-1} = \sigma^{-2} \mathbf{I} - (\sigma^2 \mathbf{I} + \theta^2 \boldsymbol{\Sigma}_{s,n})^{-1}. \quad (14)$$

Since the calculation of the likelihood ratio requires the product of all observations, the optimal detector typically requires $O(n^2)$ multiplications and $O(n)$ memory for the storage of the previous samples.⁷ Next, we consider a simple quadratic detector obtained by neglecting the signal correlation, i.e., $\boldsymbol{\Sigma}_{s,n} \equiv \mathbf{I}$, and it is given by

$$\delta_{sq}(\mathbf{y}_n) = \begin{cases} 1, & \frac{1}{n} \log \left[\left(\frac{\sigma^{2n}}{(\sigma^2 + \theta^2)^n} \right)^{1/2} e^{\frac{1}{2} \mathbf{y}_n^T \hat{\mathbf{Q}}_n \mathbf{y}_n} \right] \geq 0, \\ 0, & \text{otherwise,} \end{cases} \quad (15)$$

where

$$\hat{\mathbf{Q}}_n = \frac{\theta^2}{\sigma^2(\sigma^2 + \theta^2)} \mathbf{I}. \quad (16)$$

The test statistic in this case can be rewritten as

$$T_{sq,n} = \frac{1}{2} \log \frac{\sigma^2}{\sigma^2 + \theta^2} + \frac{\theta^2}{2n\sigma^2(\sigma^2 + \theta^2)} \sum_{i=1}^n y_i^2. \quad (17)$$

Thus, the simple quadratic detector requires $O(n)$ multiplications and one storage for accumulation.

We also consider a banded-quadratic detector structure which has intermediate complexity between the optimal and the simple quadratic detector, similar to that considered by Poor and Chang.^{7,8} Since the determinants of the two matrices $\boldsymbol{\Sigma}_{0,n}$ and $\boldsymbol{\Sigma}_{1,n}$ can be computed off line for the optimal detector (12, 13) when the signal correlation structure is known beforehand, the main complexity results from the calculation of the quadratic term based on observations. Thus, a class of detectors with intermediate complexity is given by

$$\delta_{b,m}(\mathbf{y}_n) = \begin{cases} 1, & \frac{1}{n} \log L_n^{(b,m)}(\mathbf{y}_n) \geq 0, \\ 0, & \text{otherwise,} \end{cases} \quad (18)$$

where

$$L_n^{(b,m)}(\mathbf{y}_n) = \left(\frac{|\boldsymbol{\Sigma}_{0,n}|}{|\boldsymbol{\Sigma}_{1,n}|} \right)^{1/2} e^{\frac{1}{2} \mathbf{y}_n^T \tilde{\mathbf{Q}}_n^{(m)} \mathbf{y}_n}, \quad (19)$$

and $\tilde{\mathbf{Q}}_n^{(m)}$ is a banded $n \times n$ symmetric positive-definite Toeplitz matrix with bandwidth $(2m + 1)$, i.e.,

$$\tilde{\mathbf{Q}}_n^{(m)} = \begin{bmatrix} b_0 & b_1 & \cdots & b_m & 0 & \cdots & \cdots \\ b_1 & b_0 & b_1 & \ddots & b_m & 0 & \cdots \\ \vdots & b_1 & b_0 & b_1 & \ddots & b_m & 0 \\ b_m & \ddots & \ddots & \ddots & \ddots & \ddots & b_m \\ 0 & b_m & \ddots & b_1 & b_0 & b_1 & \vdots \\ \vdots & 0 & b_m & \ddots & b_1 & b_0 & b_1 \\ \vdots & \vdots & 0 & b_m & \cdots & b_1 & b_0 \end{bmatrix}. \quad (20)$$

Here, the values of b_l , $l = 0, 1, \dots, m$ ($b_{-l} = b_l$) need to be properly determined for optimal performance. Let the discrete-time Fourier transform of the finite sequence $\{b_{-m}, b_{-m+1}, \dots, b_{-1}, b_0, b_1, \dots, b_m\}$ be $g_m(\omega)$, i.e.,

$$g_m(\omega) = b_0 + 2 \sum_{l=1}^m b_l \cos(l\omega), \quad 0 \leq \omega \leq 2\pi. \quad (21)$$

Then, the eigenvalues of $\tilde{\mathbf{Q}}_n^{(m)}$ converge to uniform samples, $\{g_m(\frac{2\pi k}{n})\}_{k=0,1,\dots,n-1}$, of $g_m(\omega)$ as n increases since $\tilde{\mathbf{Q}}_n^{(m)}$ is Toeplitz.⁹

3.1. Error Exponent and ARE

The false alarm probability, $\alpha_n^{(\delta)}$, and the miss probability, $\beta_n^{(\delta)}$, for a particular detector δ are defined as

$$\alpha_n^{(\delta)} \triangleq \Pr\{\delta(\mathbf{y}_n) = 1 | H_0\}, \quad (22)$$

$$\beta_n^{(\delta)} \triangleq \Pr\{\delta(\mathbf{y}_n) = 0 | H_1\}. \quad (23)$$

In general, these probabilities decay exponentially as n increases without bound, and the decay rate is given by Theorem 2.2. Thus, we have

$$E_0(\delta) \triangleq - \lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha_n^{(\delta)} = \inf_{z \in [0, \infty)} I_0^{(\delta)}(z), \quad (24)$$

$$E_1(\delta) \triangleq - \lim_{n \rightarrow \infty} \frac{1}{n} \log \beta_n^{(\delta)} = \inf_{z \in (-\infty, 0)} I_1^{(\delta)}(z), \quad (25)$$

where $I_j(\delta)(z)$, $j = 0, 1$, is defined as (5) with limiting cumulant moment generating function $\Lambda_j^{(\delta)}(t)$ corresponding to the considered detector and hypothesis. The error exponent or the exponential decay rate of the average error probability for the detector δ is given by

$$\begin{aligned} E(\delta) &\triangleq \lim_{n \rightarrow \infty} -\frac{1}{n} \log P_{e,n}^{(\delta)} = \lim_{n \rightarrow \infty} -\frac{1}{n} \log(\pi_0 \alpha_n^{(\delta)} + \pi_1 \beta_n^{(\delta)}), \\ &= \min\{E_0(\delta), E_1(\delta)\}. \end{aligned} \quad (26)$$

Hence, we have asymptotically

$$P_{e,n}^{(\delta)} \sim e^{-nE(\delta)}. \quad (27)$$

Eq. (27) provides an asymptotic criterion for the comparison of two detectors.⁴ The efficiency of $\{\delta_1\}$ relative to $\{\delta_2\}$ for sample size n is defined to the ratio n_2/n , where n_2 is the smallest number of samples such that $P_{e,n_2}^{(\delta_2)} \leq P_{e,n}^{(\delta_1)}$.¹ Thus, the asymptotic efficiency of a detector δ_1 relative to another detector δ_2 from the LDP is defined as the ratio between the two error exponents:

$$\text{ARE}_{\delta_1, \delta_2} \triangleq \frac{E(\delta_1)}{E(\delta_2)}. \quad (28)$$

Now let us consider the rate function for each detector under consideration. For the simple quadratic detector $\delta_{sq}(\mathbf{y}_n)$ the calculation of the rate function under each hypothesis is straightforward from Theorem 2.2. Applying Theorem 2.2 to (17), we have

$$\Lambda_0^{(\delta_{sq})}(t) = \frac{t}{2} \log \frac{\sigma^2}{\sigma^2 + \theta^2} - \frac{1}{2} \log \left(1 - t \frac{\theta^2}{\sigma^2 + \theta^2} \right), \quad (29)$$

$$\Lambda_1^{(\delta_{sq})}(t) = \frac{t}{2} \log \frac{\sigma^2}{\sigma^2 + \theta^2} - \frac{1}{4\pi} \int_0^{2\pi} \log \left(1 - t \frac{\theta^2(\sigma^2 + \theta^2 f_s(\omega))}{\sigma^2(\sigma^2 + \theta^2)} \right) d\omega, \quad (30)$$

where $f_s(\omega)$ is the spectrum of the signal and the range of t is defined for each case so that the term in the logarithmic function is strictly positive.

For the optimal detector $\delta_o(\cdot)$ the test statistic is given by

$$T_{o,n} = \frac{1}{n} \log \left[\left(\frac{|\boldsymbol{\Sigma}_{0,n}|}{|\boldsymbol{\Sigma}_{1,n}|} \right)^{1/2} e^{\frac{1}{2} \mathbf{y}_n^T \mathbf{Q}_n \mathbf{y}_n} \right]. \quad (31)$$

In this case the rate function is obtained by a whitening transform. Let the eigendecomposition of the signal covariance matrix $\boldsymbol{\Sigma}_{s,n}$ be

$$\boldsymbol{\Sigma}_{s,n} = \mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^T = \mathbf{U} \text{diag}(\lambda_1^{(n)}, \dots, \lambda_n^{(n)}) \mathbf{U}^T, \quad (32)$$

where \mathbf{U} is an orthogonal matrix. Then, the eigendecomposition of \mathbf{Q}_n is given by

$$\begin{aligned} \mathbf{Q}_n &= \mathbf{U} \mathbf{S} \mathbf{U}^T, \\ &= \mathbf{U} \text{diag} \left(\frac{\theta^2 \lambda_1^{(n)}}{\sigma^2(\sigma^2 + \theta^2 \lambda_1^{(n)})}, \dots, \frac{\theta^2 \lambda_n^{(n)}}{\sigma^2(\sigma^2 + \theta^2 \lambda_n^{(n)})} \right) \mathbf{U}^T, \end{aligned} \quad (33)$$

and

$$\mathbf{y}_n^T \mathbf{Q}_n \mathbf{y}_n = \|\bar{\mathbf{y}}_n\|^2 = \sum_{i=1}^n \bar{y}_i^2, \quad (34)$$

where $\bar{\mathbf{y}}_n = \mathbf{S}^{1/2} \mathbf{U}^T \mathbf{y}_n$ and \bar{y}_i is the i^{th} element of $\bar{\mathbf{y}}_n$. Thus, the test statistic is given by

$$T_{o,n} = \frac{1}{2n} \sum_{i=1}^n \log \frac{\sigma^2}{\sigma^2 + \theta^2 \lambda_i^{(n)}} + \frac{1}{2n} \sum_{i=1}^n \bar{y}_i^2.$$

By Theorems 2.2 and 2.4, we have

$$\Lambda_0^{(\delta_o)}(t) = \frac{t}{4\pi} \int_0^{2\pi} \log \frac{\sigma^2}{\sigma^2 + \theta^2 f_s(\omega)} d\omega - \frac{1}{4\pi} \int_0^{2\pi} \log \left(1 - t \frac{\theta^2 f_s(\omega)}{\sigma^2 + \theta^2 f_s(\omega)} \right) d\omega, \quad (35)$$

$$\Lambda_1^{(\delta_o)}(t) = \frac{t}{4\pi} \int_0^{2\pi} \log \frac{\sigma^2}{\sigma^2 + \theta^2 f_s(\omega)} d\omega - \frac{1}{4\pi} \int_0^{2\pi} \log \left(1 - t \frac{\theta^2 f_s(\omega)}{\sigma^2} \right) d\omega. \quad (36)$$

The range of t is again defined for each case so that the term in the logarithmic function is positive. For the optimal case the rate function for the quadratic part has also been derived by several other authors, e.g., Chamberland.¹⁷

For the banded quadratic detector the test statistic is given by

$$T_{b,n}^{(m)} = \frac{1}{n} \log \left[\left(\frac{|\boldsymbol{\Sigma}_{0,n}|}{|\boldsymbol{\Sigma}_{1,n}|} \right)^{1/2} e^{\frac{1}{2} \mathbf{y}_n^T \tilde{\mathbf{Q}}_n^{(m)} \mathbf{y}_n} \right] = \frac{1}{2n} \sum_{i=1}^n \log \frac{\sigma^2}{\sigma^2 + \theta^2 \tilde{\lambda}_i^{(n)}} + \frac{1}{2n} \mathbf{y}_n^T \tilde{\mathbf{Q}}_n^{(m)} \mathbf{y}_n, \quad (37)$$

where $\tilde{\mathbf{Q}}_n^{(m)}$ is defined in (20). By Lemma 2.3, the cumulant generating function for the quadratic part under the hypothesis j is given by

$$\log \mathbb{E}_j \{ e^{t \frac{1}{2} \mathbf{y}_n^T \tilde{\mathbf{Q}}_n^{(m)} \mathbf{y}_n} \} = -\frac{1}{2} \sum_{i=1}^n \log \left(1 - t \tilde{\lambda}_{ij}^{(n)} \right), \quad j = 0, 1, \quad (38)$$

for all $t < 1/(\max_i \tilde{\lambda}_{ij}^{(n)})$, where $\{\tilde{\lambda}_{ij}^{(n)}, i = 1, \dots, n\}$ are the eigenvalues of $\tilde{\mathbf{Q}}_n^{(m)} \boldsymbol{\Sigma}_{j,n}$, and $\boldsymbol{\Sigma}_{j,n}$ ($j = 0, 1$) is defined in (10). Because of the Toeplitz structure of $\tilde{\mathbf{Q}}_n^{(m)}$ and $\boldsymbol{\Sigma}_{j,n}$, it follows that⁹

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n h(\tilde{\lambda}_{ij}^{(n)}) = \frac{1}{2\pi} \int_0^{2\pi} h(g_m(\omega) f_j(\omega)) d\omega \quad (39)$$

for any continuous function $h(\cdot)$, where $f_j(\omega)$ is the spectrum of the observation process $\{y_i\}$ under the hypothesis j ($j = 0, 1$). Thus, the rate function for the banded-quadratic detector is given by

$$\Lambda_0^{(\delta_{b,m})}(t) = \frac{t}{4\pi} \int_0^{2\pi} \log \frac{\sigma^2}{\sigma^2 + \theta^2 f_s(\omega)} d\omega - \frac{1}{4\pi} \int_0^{2\pi} \log (1 - t\sigma^2 g_m(\omega)) d\omega, \quad (40)$$

$$\Lambda_1^{(\delta_{b,m})}(t) = \frac{t}{4\pi} \int_0^{2\pi} \log \frac{\sigma^2}{\sigma^2 + \theta^2 f_s(\omega)} d\omega - \frac{1}{4\pi} \int_0^{2\pi} \log (1 - t(\sigma^2 + \theta^2 f_s(\omega))g_m(\omega)) d\omega, \quad (41)$$

where $g_m(\omega)$ is defined in (21) and the range of t is defined properly in each case.

4. EXAMPLES AND NUMERICAL RESULTS

We now consider some signal examples and investigate the relative performance of the detectors in the previous section as a function of various parameters such as correlation strength and SNR via the asymptotic relative efficiency defined in (28). In particular, we consider Gauss-Markov signals and triangularly correlated signals. Except for some simple cases such as autoregressive signals without additive noise¹⁴ it is difficult to obtain closed-form expressions for the rate functions in the previous section. Thus, we evaluate the rate and ARE by numerical evaluation of the error exponent.

4.1. Gauss-Markov Signal

We first consider the stationary Gauss-Markov signal for which the correlation is given by

$$\mathbb{E}\{S_0 S_k\} = a^{|k|}, \quad k = 0, \pm 1, \pm 2, \dots, \quad (0 \leq a < 1), \quad (42)$$

and the spectrum is given by the Poisson kernel:

$$f_s(\omega) = \frac{1 - a^2}{1 - 2a \cos \omega + a^2}. \quad (43)$$

Fig. 1 (a) shows the error exponent for the false alarm and miss probabilities for the optimal and simple quadratic detectors as a function of the correlation strength a at 10 dB SNR. It is seen that the error exponent $E_0(\delta_{sq})$ for the false alarm probability of the simple quadratic detector is independent of the correlation strength and is equal to the maximum value of the error exponent of the optimal detector achieved by independent signal[†] ($a = 0$). This is easily seen by the logarithmic generating function (29) which does not depend on the signal spectrum. However, the error exponent $E_1(\delta_{sq})$ for the miss probability is less than that of the false alarm probability for all values of a , and decreases to zero as the signal correlation becomes strong ($a \rightarrow 1$). Thus, the error exponent for the average error probability is determined by that of the miss probability for the simple quadratic detector. On the other hand, the error exponents for the false alarm and miss probabilities are the same, i.e., $E_0(\delta_o) = E_1(\delta_o)$, for the optimal detector with equal prior probabilities, i.e., zero threshold in (12). In this case, the minimum in (26) is attained and the error exponent is the Chernoff information between the two distributions under the hypotheses (9). Note that the error exponent for the miss probability of the simple quadratic detector is smaller than that of the optimal detector for $0 < a < 1$. So, the error exponent of the simple quadratic detector is smaller than that of the optimal detector even if the simple quadratic detector performs better than the optimal detector for the false alarm probability. From the detector structure (17) one can see that the simple quadratic detector is optimized for the detection of the false alarm event regardless of the signal correlation, thereby sacrificing the performance for correct detection, while the optimal detector optimizes the test statistic so that it can perform equally well for both of the false alarm and miss events.

Fig. 1 (b) shows the ARE of the simple quadratic detector to the optimal detector as a function of correlation strength a for several values of SNR (0, 10, 20, 30 dB). It is seen that at weak correlation the simple quadratic detector performs as well as the optimal detector for all the values of SNR. It is also seen that the ARE decreases

[†]This is not the case when the SNR is low. At low SNR the maximum value of the error exponent for the optimal detection is achieved at some correlation value $0 < a < 1$.²⁰

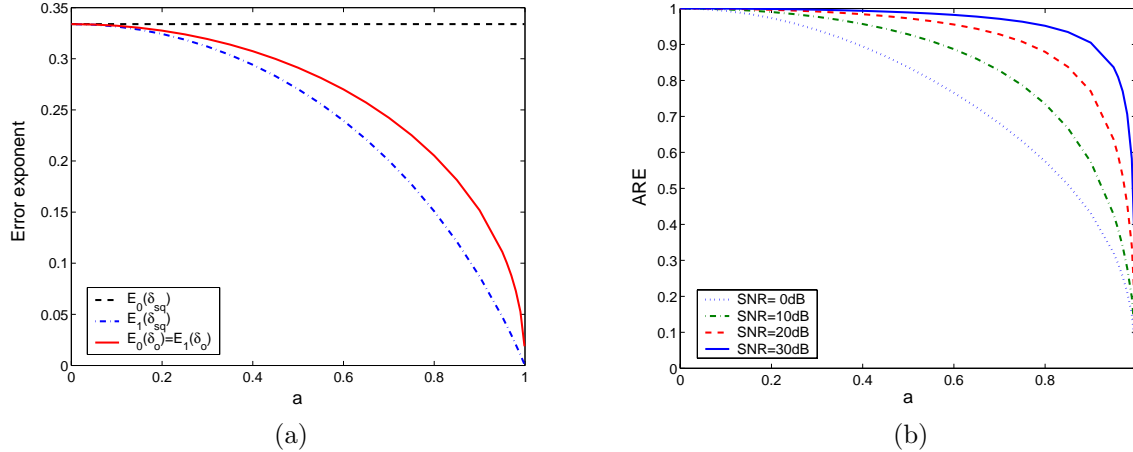


Figure 1. Optimal and simple quadratic detectors (Gauss-Markov signal): (a) error exponent, $E_j(\delta)$, $j = 0, 1$, as a function of correlation strength a (SNR=10dB) and (b) ARE as a function of correlation strength a for SNR = 0, 10, 20, 30 dB.

to zero eventually as the correlation becomes strong ($a \rightarrow 1$). This is because for the perfectly correlated signal ($a = 1$) the optimal test statistic is in form of $(\sum_{i=1}^n y_i)^2$ which uses the perfect signal correlation and adds the signal component coherently before taking the magnitude by squaring.²⁰ On the other hand, the test statistic (17) for the simple quadratic detector neglects this correlation entirely. It is seen that the range of correlation values over which the simple quadratic detector performs as well as the optimal detector increases as SNR increases. Note that at an SNR of 30 dB the simple quadratic detector performs as well as the optimal detector through almost the whole range of correlation except the very highly correlated case ($0.9 < a \leq 1$). The behavior of ARE as a function of SNR is summarized in the following proposition.

PROPOSITION 4.1. *The ARE of the simple quadratic detector (15) to the optimal detector (12) increases to unity for any bounded spectrum $f_s(\omega)$ as SNR increases without bound.*

Proof: Since the error exponent for the simple quadratic detector is determined by the miss probability and the optimal detector has the same error exponent for the false alarm and miss probabilities, this can be shown via the cumulant generating functions (30, 36) for the two detectors. For any bounded spectrum we have

$$f_s(\omega) \leq M, \quad \forall 0 \leq \omega \leq 2\pi, \quad (44)$$

for some $M > 0$. So, we have for the second term in (30), as $\theta^2 \rightarrow \infty$,

$$\frac{\theta^2(\sigma^2 + \theta^2 f_s(\omega))}{\sigma^2(\sigma^2 + \theta^2)} \rightarrow \frac{\theta^2 f_s(\omega)}{\sigma^2}, \quad (45)$$

which is the corresponding term in (36). For the first terms in (30) and (36) we have

$$\log \frac{\sigma^2}{\sigma^2 + \theta^2} \rightarrow \log \frac{\sigma^2}{\theta^2} \quad (46)$$

$$\int_0^{2\pi} \log \frac{\sigma^2}{\sigma^2 + \theta^2 f_s(\omega)} d\omega \rightarrow \int_0^{2\pi} \log \frac{\sigma^2}{\theta^2 f_s(\omega)} d\omega = \log \frac{\sigma^2}{\theta^2}, \quad (47)$$

since $\int_0^{2\pi} \log f_s(\omega) d\omega = 0$ because of the para-Hermitian conjugacy of the spectral factorization of $f_s(\omega) = L(z)L^*(\frac{1}{z^*})|_{z=e^{j\omega}}$. Thus, the two rate functions for the simple quadratic and the optimal detectors converge as $\theta^2 \rightarrow \infty$. ■

For the spectrum (43) we have bounded spectrum for any fixed value of a ($0 \leq a < 1$), which explains the behavior of the ARE in Fig. 1 (b) as SNR increases.

4.2. Triangularly Correlated Signal

Next we consider the stationary signal with triangular correlation, i.e.,

$$\mathbb{E}\{S_0 S_k\} = \begin{cases} 1 - |k|/M, & |k| < M \\ 0, & |k| \geq M \end{cases} \quad (48)$$

where $M > 0$ is the correlation length of the signal. The spectrum of the signal is given by the M th Fejék kernel⁷:

$$f_s(\omega) = \frac{1}{M} \left(\frac{\sin(M\omega/2)}{\sin(\omega/2)} \right)^2, \quad 0 \leq \omega \leq 2\pi. \quad (49)$$

Fig. 2 (a) shows the error exponent for the false alarm and miss probabilities for the optimal and simple quadratic detectors as a function of the correlation width M at 10 dB SNR for the triangularly correlated signal. Similar relative behavior to that in the Gauss-Markov signal case is observed. It is worth noticing that the error exponents for the two detectors decay sharply near $M = 1$ as the correlation length M increases, and the decay is mild as M further increases.

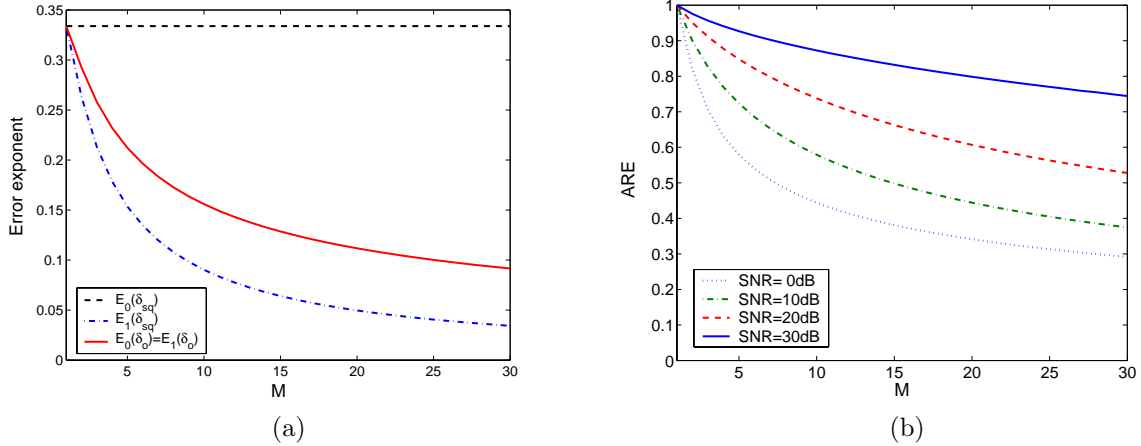


Figure 2. Optimal and simple quadratic detectors (triangularly correlated signal): (a) error exponent, $E_j(\delta)$, $j = 0, 1$, as a function of correlation strength a (SNR=10dB) and (b) ARE as a function of correlation strength a for SNR = 0, 10, 20, 30 dB.

Fig. 2 (b) shows the ARE of the simple quadratic detector to the optimal detector as a function of correlation strength a for the same values of SNR as in the Gauss-Markov case. It is seen that the ARE increases as SNR increases as expected from Proposition 4.1. However, at an SNR of 30 dB there exists noticeable performance degradation for the simple quadratic detector compared with Fig. 1 (b) for a wide range of the correlation length M .

4.3. Banded Quadratic Detector

We here provide some necessary conditions for the optimal $\tilde{\mathbf{Q}}_n^{(m)}$ in (19) and evaluate the performance of the banded quadratic detector. The test statistic (37) has two different limits (as $n \rightarrow \infty$) under the two hypotheses, and they are given by

$$\bar{T}_0^{(m)} \triangleq \lim_{n \rightarrow \infty} \{T_{b,n}^{(m)} | H_0\} = \frac{1}{4\pi} \int_0^{2\pi} \log \frac{\sigma^2}{\sigma^2 + \theta^2 f_s(\omega)} d\omega + \frac{1}{4\pi} \int_0^{2\pi} \sigma^2 g_m(\omega) d\omega, \quad (50)$$

$$\bar{T}_1^{(m)} \triangleq \lim_{n \rightarrow \infty} \{T_{b,n}^{(m)} | H_1\} = \frac{1}{4\pi} \int_0^{2\pi} \log \frac{\sigma^2}{\sigma^2 + \theta^2 f_s(\omega)} d\omega + \frac{1}{4\pi} \int_0^{2\pi} (\sigma^2 + \theta^2 f_s(\omega)) g_m(\omega) d\omega. \quad (51)$$

The first term in each equation is by applying Theorem 2.4, and the second term follows from the law of large numbers and $\frac{1}{n} \mathbb{E}_j \{ \mathbf{y}_n^T \tilde{\mathbf{Q}}_n^{(m)} \mathbf{y}_n \} = \frac{1}{n} \text{tr} \{ \tilde{\mathbf{Q}}_n^{(m)} \boldsymbol{\Sigma}_{j,n} \} = \frac{1}{n} \sum_{i=1}^n \tilde{\lambda}_{ij}^{(n)}$ (to which (39) is applied) since the trace of a

matrix is the sum of its eigenvalues. From (50, 51) we have $\bar{T}_0^{(m)} < \bar{T}_1^{(m)}$ for any $\theta > 0$ and a signal spectrum which is not identically zero. One necessary condition for the optimal $g_m(\omega)$ is given by

$$\bar{T}_0^{(m)} < \tau (= 0) < \bar{T}_1^{(m)}. \quad (52)$$

Otherwise, the error exponent $E(\delta_{b,m})$ is zero and the average error probability of the banded-quadratic detector decays at subexponential rate as n increases. For example, if $\bar{T}_0^{(m)} > 0$, then $E_0(\delta_{b,m}) = \inf_{z \in [0, \infty)} I_0^{\delta_{b,m}}(z) = 0$ since $I_0^{\delta_{b,m}}(\bar{T}_0^{(m)}) = 0$. Similarly, we have $E_1(\delta_{b,m}) = \inf_{z \in (-\infty, 0)} I_1^{\delta_{b,m}}(z) = 0$ if $\bar{T}_1^{(m)} < 0$. Thus, in the case of $m = 0$ we have $g_m(\omega) = b_0$ and it is seen from (51) that the optimal b_0 is positive (otherwise, $\bar{T}_1^{(0)} < 0$), which is consistent with our assumption of the positive-definiteness of $\tilde{\mathbf{Q}}_n^{(m)}$. In general, it is easy to see that well chosen b_0, \dots, b_m satisfy the condition (52) since the first terms in (50, 51) are equivalent and negative. When (52) is satisfied, it is known that the infimum for the rate function is achieved at the decision threshold,¹¹ i.e.,

$$E_0(\delta_{b,m}) = \inf_{z \in [0, \infty)} I_0^{\delta_{b,m}}(z) = I_0^{\delta_{b,m}}(0) = \sup_{-\infty < t < \inf_{0 \leq \omega \leq 2\pi} \left\{ \frac{1}{\sigma^2 g_m(\omega)} \right\}} \{-\Lambda_0^{\delta_{b,m}}(t)\}, \quad (53)$$

$$E_1(\delta_{b,m}) = \inf_{z \in (-\infty, 0)} I_1^{\delta_{b,m}}(z) = I_1^{\delta_{b,m}}(0) = \sup_{-\infty < t < \inf_{0 \leq \omega \leq 2\pi} \left\{ \frac{1}{(\sigma^2 + \theta^2 f_s(\omega)) g_m(\omega)} \right\}} \{-\Lambda_1^{\delta_{b,m}}(t)\}, \quad (54)$$

where $\Lambda_0^{\delta_{b,m}}(t)$ and $\Lambda_1^{\delta_{b,m}}(t)$ are given by (40) and (41), respectively, and the optimal values of t for (53) and (54) are given by solving

$$\frac{1}{4\pi} \int_0^{2\pi} \log \frac{\sigma^2}{\sigma^2 + \theta^2 f_s(\omega)} d\omega + \frac{1}{4\pi} \int_0^{2\pi} \frac{\sigma^2 g_m(\omega)}{1 - t_0^* \sigma^2 g_m(\omega)} d\omega = 0 \quad (55)$$

and

$$\frac{1}{4\pi} \int_0^{2\pi} \log \frac{\sigma^2}{\sigma^2 + \theta^2 f_s(\omega)} d\omega + \frac{1}{4\pi} \int_0^{2\pi} \frac{(\sigma^2 + \theta^2 f_s(\omega)) g_m(\omega)}{1 - t_1^* (\sigma^2 + \theta^2 f_s(\omega)) g_m(\omega)} d\omega = 0, \quad (56)$$

respectively. Thus, the optimal $g_m(\omega)$ for given m , SNR and signal spectrum is obtained from the following optimization problem:

$$g_m^*(\omega) = \arg \max_{b_0, \dots, b_m} \left\{ \min \{ I_0^{\delta_{b,m}}(0), I_1^{\delta_{b,m}}(0) \} \right\} \quad (57)$$

under the constraint (52). A closed-form expression for (57) seems difficult to obtain in general cases. However, (52-56) facilitate numerical approaches to the optimization problem, and a procedure using grid search is summarized in Fig. 3.

We considered the Gauss-Markov signal (43) and evaluated the banded-quadratic detector with $m = 1$ which corresponds to the case that each sensor requires the information only from a neighboring sensor in a wireless sensor network setup. Fig. 4 (a) shows the error exponents $E_0(\delta_{b,1})$ and $E_1(\delta_{b,1})$ of the banded-quadratic detector optimized using the algorithm shown in Fig. 3 for each value of a at 10 dB SNR. Fig. 4 (b) shows the corresponding ARE of the banded-quadratic detector to the optimal detector. Surprisingly, it is seen that optimal performance is almost achieved with only $m = 1$ for a SNR of 10 dB.

5. CONCLUSIONS

We have considered the relative performance of several quadratic detectors for Gaussian signals in Gaussian noise under a Bayesian formulation. Using the large deviations principle, a general form of the rate function for the simple quadratic detector, optimal detector, and banded-quadratic detector has been provided using the signal spectrum. For the examples of Gauss-Markov and triangularly correlated signals we have evaluated the error exponents for the false alarm and miss probabilities and the ARE for the average error probability. We have also investigated the effects of SNR on the relative performance. The asymptotic efficiency of the simple quadratic detector relative to the optimal detector converges to unity as SNR increases without bound for any bounded signal spectrum. At high SNR the simple quadratic detector performs as well as the optimal detector for a wide range of correlation values and the banded-quadratic detector effectively achieves the optimal performance with much lower complexity.

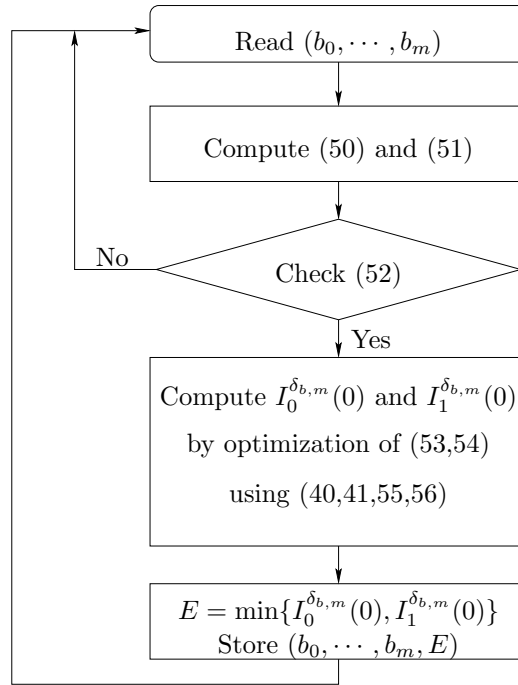


Figure 3. An optimization algorithm (grid search).

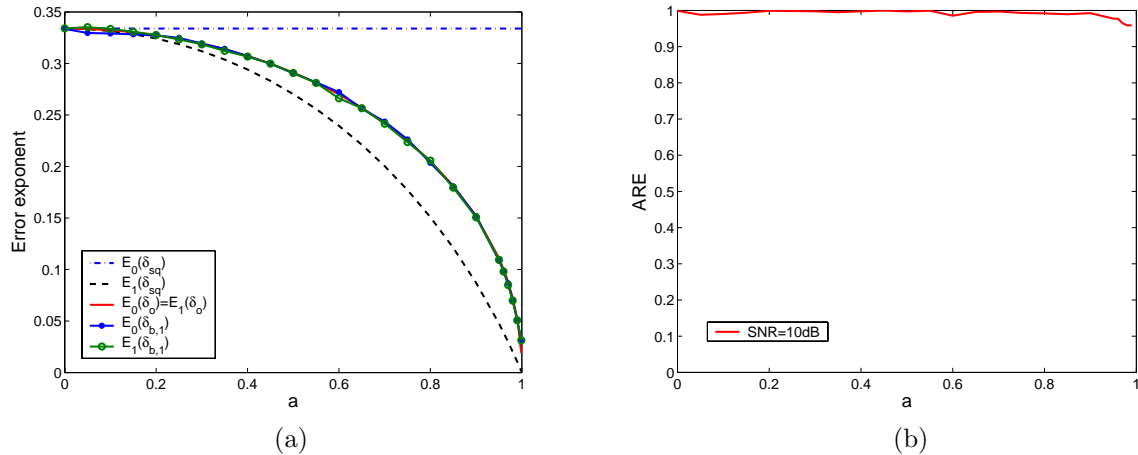


Figure 4. Banded-quadratic detector (Gauss-Markov signal): (a) error exponent, $E_j(\delta)$, $j = 0, 1$, as a function of correlation strength a for SNR = 10 dB and (b) ARE as a function of correlation strength a with the optimized $g_m(\omega)$ for SNR = 10 dB.

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