Capacity of the Linear Time-Invariant Gaussian Relay Channel

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***Abstract*— In this paper, the Gaussian relay channel with linear time-invariant relay filtering is considered. Based on spectral theory for stationary processes, the maximum achievable rate of this subclass of Gaussian relay operation is obtained in finite-letter characterization. The maximum rate of this subclass can be achieved by dividing the overall frequency band (or the overall time interval alternatively) into at most eight segments and by making the relay behave as an instantaneous amplify-and-forward relay at each segment. Numerical results are provided to evaluate the performance of LTI relaying, and the numerical results show that LTI relaying does not increase the rate considerably over the instantaneous amplify-and-forward relay in flat-fading Gaussian relay channels.**

I. Introduction

The relay channel problem is one of the classical problems in information theory, and still the capacity of this three node network is not exactly known. However, many ingenious coding strategies including decode-and-forward, compress-and-forward, etc. beyond the simple instantaneous amplify-and-forward (IAF) scheme have been developed [1], [2]. Recently, El Gamal et al. proposed a more advanced linear scheme for relay channels based on linear processing at the relay to compromise the complexity and performance between the complicated coding strategies and IAF [3], and showed that the scheme could perform well in certain cases by giving an example. Although the capacity for frequency-division linear relaying was obtained in their work, the general linear relay case was not explored fully, and the capacity for the general linear relay channel was not obtained; the general linear problem is a sequence of non-convex optimization problems and seemingly intractable [3] except the simple case of one-tap IAF [4]. To circumvent such difficulty, in [5] we considered more tractable and practical linear time-invariant (LTI) relaying, and proposed an efficient joint design algorithm for source and relay filters for general inter-symbol interference (ISI) Gaussian relay channels. However, a performance bound for LTI relaying was not obtained in [5]. In this paper, we derive the maximum achievable rate of LTI relaying in finite-letter characterization, based on results from spectral theory [6], [7], [8] and a technique similar to that used in [3]. The obtained result here provides new insights into the structure and performance of optimal linear relay processing.

Notations: We will make use of standard notational conventions. Vectors and matrices are written in boldface with matrices in capitals. All vectors are column vectors. For scalar a, a^* denotes its complex conjugate. For matrix $\mathbf{A}$, $\mathbf{A}^T$, $\mathbf{A}^H$ and tr$(\mathbf{A})$ indicate the transpose, Hermitian transpose and trace of $\mathbf{A}$, respectively. $\mathbf{I}_n$ stands for the identity matrix of size n (the subscript is omitted when unnecessary). The notation $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ means that $\mathbf{x}$ is Gaussian distributed with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. $\mathbb{E}\{\cdot\}$ denotes the expectation. $\mathbb{R}$ and $\mathbb{C}$ are the sets of reals and complex numbers, respectively. $\iota = \sqrt{-1}$.

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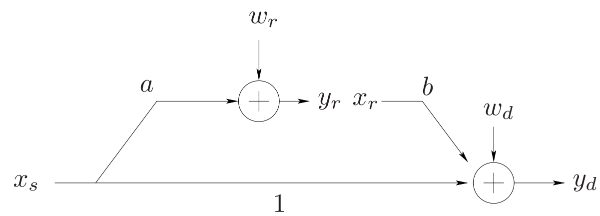


Fig. 1. System model

II. System Model and Background

We consider the general additive white Gaussian noise (AWGN) relay channel in Fig. 1. Here, x_s is the transmitted symbol at the source; x_r and y_r are the transmitted and received symbols at the relay, respectively; and y_d is the received symbol at the destination. We assume that the channel coefficients from the source to the destination, from the source to the relay, and from the relay to the destination are 1, a, and b, respectively. Then, the received signals at the relay and the destination at the i-th symbol time are given by

$$\begin{aligned} y_r[i] &= a x_s[i] + w_r[i], \qquad \text{and} \\ y_d[i] &= x_s[i] + b x_r[i] + w_d[i], \end{aligned}$$

respectively, where $w_s[i]$ and $w_r[i]$ are independent and both are from $\mathcal{N}(0, \sigma^2)$. The source and relay have maximum available per-symbol average power P and γP, respectively, for some $\gamma > 0$.

Here, we introduce the *Toeplitz distribution theorem* for our later development.

Theorem 1: [6] Let $\{r_k^y := \mathbb{E}\{y_n y_{n-k}^*\}\}$ be an absolutely summable autocovariance sequence of a stationary process $\{y_n\}$; let $\boldsymbol{\Sigma}_n^y = [r_{i-j}^y]_{i,j=1}^n$ be its Toeplitz covariance matrix; let $f^y(\omega) := \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} r_k^y e^{-\iota k \omega}$ be the spectrum of $\{y_n\}$; and let $\{\zeta_i^{(n)}\}$ be the eigenvalues of $\boldsymbol{\Sigma}_n^y$. Then,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} g(\zeta_i^{(n)}) = \frac{1}{2\pi}\int_0^{2\pi} g(f^y(\omega))d\omega \qquad (1)$$

for any continuous function $g(\cdot)$.

III. Linear Time-Invariant Relaying

A. General LTI relaying

The general (possibly noncausal) linear processing at the relay is given by $x_r[i] = \sum_{j=-\infty}^{\infty} h_{ij} y_r[j]$ for arbitrary linear combination coefficients h_{ij}. However, such linear processing requires time-varying filtering at the relay and is not readily realizable. Thus, in this paper we restrict ourselves to the case of LTI filtering at the relay. In this case, the relay output is given by

$$x_r[i] = \sum_{j=-\infty}^{\infty} h_j y_r[i-j], \quad (2)$$

where $[\cdots, h_{-1}, h_0, h_1, h_2, \cdots]$ is the (possibly noncausal) LTI impulse response of the relay filter which is assumed to be stable, i.e., $\sum_{j=-\infty}^{+\infty} |h_j| < \infty$. Thus, the frequency response $H(\omega)$ of the relay filter is well defined as $H(\omega) = (1/2\pi)\sum_{j=-\infty}^{\infty} h_j e^{-\iota j\omega}$. Note that the frequency response $H(\omega)$ is complex in general since $\{h_j\}$ is arbitrary except being stable. (2) can be written in vector form as

$$\mathbf{x}_n^r = \mathbf{H}_n \mathbf{y}_n^r,$$

where $\mathbf{x}_n^r = [x_r[1], x_r[2], \cdots, x_r[n]]^T$, $\mathbf{y}_n^r = [y_r[1], y_r[2], \cdots, y_r[n]]^T$, and

$$\mathbf{H}_n = \begin{bmatrix} h_0 & h_{-1} & \cdots & h_{-n+1} \\ h_1 & h_0 & \cdots & \\ \vdots & \ddots & \ddots & h_{-1} \\ h_{n-1} & \cdots & h_1 & h_0 \end{bmatrix}$$

With the LTI filtering relay, the overall channel from the source to the destination becomes a Gaussian ISI channel, and stationary Gaussian input distribution is sufficient to achieve the capacity [9, pp.407-430]. Thus, we assume stationary Gaussian input distribution hereafter:

$$\mathbf{x}_n^s = [x_s[1], x_s[2], \cdots, x_s[n]]^T \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_n^s),$$

where $\mathbf{\Sigma}_n^s$ is Hermitian and Toeplitz by the stationary of the input process. Then, the power constraints for the source and relay are respectively given by

$$\begin{aligned} \mathrm{tr}(\mathbf{\Sigma}_n^s) &\le nP, \qquad \text{and} \\ \mathbb{E}\{\mathrm{tr}(\mathbf{H}_n\mathbf{y}_n^r(\mathbf{H}_n\mathbf{y}_n^r)^H)\} &= \mathrm{tr}(\mathbf{H}_n(a^2\mathbf{\Sigma}_n^s + \sigma^2\mathbf{I})\mathbf{H}_n^H) \le n\gamma P. \end{aligned} \quad (3)$$

The received signal vector at the destination is given by

$$\mathbf{y}_n^d = \mathbf{x}_n^s + b\mathbf{x}_n^r + \mathbf{w}_n^d = (\mathbf{I} + ab\mathbf{H}_n)\mathbf{x}_n^s + b\mathbf{H}_n\mathbf{w}_n^r + \mathbf{w}_n^d,$$

where $\mathbf{y}_n^d = [y_d[1], \cdots, y_d[n]]^T$ and $\mathbf{w}_n^r, \mathbf{w}_n^d \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$. The transmission rate in this case is given by $\frac{1}{n}I(\mathbf{x}_n^s; \mathbf{y}_n^d)$

$$\begin{aligned} &= \frac{1}{2n}\log\frac{|(\mathbf{I}+ab\mathbf{H}_n)\mathbf{\Sigma}_n^s(I+ab\mathbf{H}_n)^H + \sigma^2(I+b^2\mathbf{H}_n\mathbf{H}_n^H)|}{|\sigma^2(\mathbf{I}+b^2\mathbf{H}_n\mathbf{H}_n^H)|}, \\ &= \frac{1}{2n}\log\left|\mathbf{I} + \mathbf{G}_n\mathbf{\Sigma}_n^s\mathbf{G}_n^H\right|, \end{aligned} \quad (4)$$

where $\mathbf{G}_n = \sigma^{-1}(\mathbf{I} + b^2\mathbf{H}_n\mathbf{H}_n^H)^{-1/2}(\mathbf{I} + ab\mathbf{H}_n)$. Thus, the maximum rate with LTI relaying for block size n is given by maximizing the mutual information (4) over $\mathbf{\Sigma}_n^s$ and $\mathbf{H}_n$ under the power constraints (3), and the capacity with possibly noncausal LTI relaying is given by its limit

$$C_{LTI} = \lim_{n\to\infty} \sup_{\mathbf{\Sigma}_n^s, \mathbf{H}_n} \frac{1}{n} I(\mathbf{x}_n^s; \mathbf{y}_n^d) \quad (5)$$

as $n \to \infty$, if the limit exists [3]. The capacity expression in (5) has infinite-letter characterization. In the next section, we will derive an expression for the maximum achievable rate in this LTI relaying case in *finite-letter* characterization, based on a similar technique to that used in [3] and the Toeplitz distribution theorem.

B. The capacity for LTI relaying

First, let $\mathbf{\Sigma}_n^d$ denote the covariance matrix of the noise-whitened output symbol vector at the destination in (4), i.e., $\mathbf{\Sigma}_n^d := \mathbf{I} + \mathbf{G}_n\mathbf{\Sigma}_n^s\mathbf{G}_n^H$, and let $\{\zeta_{d,i}^{(n)}, i = 1, \cdots, n\}$ be the eigenvalues of $\mathbf{\Sigma}_n^d$. The spectrum of the noise-whitened output process at the destination is simply given by [10]

$$f^d(\omega) = 1 + \frac{|1 + abH(\omega)|^2}{\sigma^2(1 + b^2|H(\omega)|^2)} f^s(\omega), \quad (6)$$

where $f^s(\omega)$ is the input spectrum and $H(\omega)$ is the frequency response of the relay filter. Also, the spectrum of the relay output is given by

$$f^r(\omega) = (a^2 f^s(\omega) + \sigma^2)|H(\omega)|^2. \quad (7)$$

Let the n uniform samples of $f^d(\omega)$ and those of $f^r(\omega)$ over $\omega \in [0, 2\pi)$ be $\{\xi_{d,i}^{(n)}, i = 1, \cdots, n\}$ and $\{\xi_{r,i}^{(n)}, i = 1, \cdots, n\}$, respectively, i.e.,

$\xi_{d,i}^{(n)} := f^d(\omega)|_{\omega=(2\pi(i-1)/n)}$ and $\xi_{r,i}^{(n)} := f^r(\omega)|_{\omega=(2\pi(i-1)/n)}$.

By (6) and (7) we have

$$\begin{aligned} \xi_{d,i}^{(n)} &= 1 + \frac{|1 + ab\lambda_i^{(n)}|^2}{\sigma^2(1 + b^2|\lambda_i^{(n)}|^2)} \mu_i^{(n)}, && (8) \\ \xi_{r,i}^{(n)} &= (a^2\mu_i^{(n)} + \sigma^2)|\lambda_i^{(n)}|^2, && (9) \end{aligned}$$

for $i = 1, \cdots, n$, where $\{\mu_i^{(n)}\}$ and $\{\lambda_i^{(n)}\}$ are the n uniform samples of the input spectrum $f^s(\omega)$ and those of the frequency response $H(\omega)$ of the relay filter, respectively, over $\omega \in [0, 2\pi)$. Note that $\{\mu_i^{(n)}\}$ are real and $\{\lambda_i^{(n)}\}$ are *complex* in general. (Hereafter, we will omit the superscript (n) for notational simplicity.) Then, we have

$$\frac{1}{n}\left| I(\mathbf{x}_s^n; \mathbf{y}_d^n) - \frac{1}{2}\sum_{i=1}^{n} \log \xi_{d,i} \right| \le \epsilon_n \quad (10)$$

for some $\epsilon_n \downarrow 0$ as $n \to \infty$, since

$$\begin{aligned} &\left| \frac{1}{n} I(\mathbf{x}_s^n; \mathbf{y}_d^n) - \frac{1}{4\pi}\int_0^{2\pi} \log(f^d(\omega))d\omega + \frac{1}{4\pi}\int_0^{2\pi} \log(f^d(\omega))d\omega \right. \\ &\left. - \frac{1}{2n}\sum_{i=1}^{n} \log \xi_{d,i} \right| \le \left| \frac{1}{n} I(\mathbf{x}_s^n; \mathbf{y}_d^n) - \frac{1}{4\pi}\int_0^{2\pi} \log(f^d(\omega))d\omega \right| \\ &\qquad + \left| \frac{1}{4\pi}\int_0^{2\pi} \log(f^d(\omega))d\omega - \frac{1}{2n}\sum_{i=1}^{n} \log \xi_{d,i} \right| \le \epsilon_n. \end{aligned} \quad (11)$$

The first inequality is obtained by the triangle inequality. The first term in the right-handed side (RHS) of the first

$$\frac{1}{2n}\sum_{i=1}^{n}\log\left(1+\frac{\mu_i}{\sigma^2}\frac{|1+ab\lambda_i|^2}{1+b^2|\lambda_i|^2}\right)-\epsilon_n \le \frac{1}{n}I(\mathbf{x}_s^n;\mathbf{y}_d^n)\le \frac{1}{2n}\sum_{i=1}^{n}\log\left(1+\frac{\mu_i}{\sigma^2}\frac{|1+ab\lambda_i|^2}{1+b^2|\lambda_i|^2}\right)+\epsilon_n \tag{12}$$

.

inequality in (11) decays to zero by Theorem 1 because $I(\mathbf{x}_s^n;\mathbf{y}_d^n) = (1/2)\log|\mathbf{\Sigma}_n^d| = (1/2)\sum_i \log\zeta_{d,i}$, $f(x) = \log x$ is continuous over $x>0$ and the eigenvalues of $\mathbf{\Sigma}_n^d$ is away from zero due to the added identity matrix. The second term in the RHS of the first inequality in (11) also decays to zero since $\frac{1}{2n}\sum_{i=1}^n \log\xi_{d,i}$ is the Riemann sum for the integral $\frac{1}{4\pi}\int_0^{2\pi}\log(f^d(\omega))d\omega$; it converges for any almost-surely continuous spectrum $f^d(\omega)$ over the domain $[0,2\pi)$. (Note that $f^d(\omega)\ge 1,\ \forall\ \omega\in[0,2\pi)$. See (6).) (10) implies (12). Similarly, the powers at the source and relay are respectively given in terms of $\{\mu_i,\lambda_i\}$ by

$$\frac{1}{n}\left|\mathrm{tr}(\mathbf{\Sigma}_n^s)-\sum_{i=1}^n\mu_i\right|\le\epsilon_n' \quad \text{and} \tag{13}$$

$$\frac{1}{n}\left|\mathrm{tr}(\mathbf{H}_n(a^2\mathbf{\Sigma}_n^s+\sigma^2\mathbf{I})\mathbf{H}_n^H)-\sum_{i=1}^n(a^2\mu_i+\sigma^2)|\lambda_i|^2\right|\le\epsilon_n'' \tag{14}$$

for some $\epsilon_n'\downarrow 0$ and $\epsilon_n''\downarrow 0$ as $n\to\infty$. By (12,13,14), for sufficiently large n, the maximum rate for LTI relaying with block size n is given by

$$\bar{R}_{LTI}^{(n)}(P,\gamma P)=\max_{\{\mu_i\},\{\lambda_i\}}\frac{1}{2n}\sum_{i=1}^n\log\left(1+\frac{\mu_i}{\sigma^2}\cdot\frac{|1+ab\lambda_i|^2}{1+b^2|\lambda_i|^2}\right)\pm\epsilon_n, \tag{15}$$

with slight abuse of the notation $\pm$, subject to the constraints $\sum_{i=1}^n\mu_i\le n(P-\epsilon_n')$, $\sum_{i=1}^n(a^2\mu_i+\sigma^2)|\lambda_i|^2\le n\gamma(P-\epsilon_n'')$ and $\mu_i\ge 0$ for $i=1,\cdots,n$.

Now let us derive $\lim_{n\to\infty}\bar{R}_{LTI}^{(n)}(P,\gamma P)$. To derive a finite-letter expression for the limit, we follow the technique used to obtain the capacity for the frequency-division linear relay channel by El Gamal et al. [3]. First, suppose that there exists $n_0\in\{1,2,\cdots,n\}$ such that $\lambda_1=\cdots=\lambda_{n_0}=0$ and assume that $\mu_i>0$ and $\lambda_i\ne 0$ for $i>n_0$ without loss of optimality. Let $\theta_0\in[0,1]$ be the portion of the total source power $n(P-\epsilon_n')$ used by $\mu_1,\cdots,\mu_{n_0}$. Then, $\sum_{i=1}^{n_0}\mu_i=\theta_0 n(P-\epsilon_n')$ and the relay does not allocate any power to these bins out of the total relay power $n\gamma(P-\epsilon_n'')$. Thus, each bin is a point-to-point channel with the same channel coefficient, and hence the optimal source power allocation is $\mu_i=\frac{\theta_0 n(P-\epsilon_n')}{n_0}$ for $i=1,\cdots,n_0$. For global optimality the Karush-Kuhn-Tucker (KKT) condition should be satisfied for the remaining variables $\{\mu_i,\lambda_i,i=n_0+1,\cdots,n\}$. For the problem (15) the Lagrangian and KKT condition are respectively given by

$$\begin{aligned}\mathcal{L} &= \frac{1}{2n}\sum_{i=n_0+1}^n\log\left(1+\frac{\mu_i}{\sigma^2}\cdot\frac{|1+ab\lambda_i|^2}{1+b^2|\lambda_i|^2}\right)+\alpha\bigg(n(P-\epsilon_n') \\ &\quad -\sum_{i=n_0+1}^n\mu_i\bigg)+\beta\bigg(n\gamma(P-\epsilon_n'')-\sum_{i=n_0+1}^n(a^2\mu_i+\sigma^2)|\lambda_i|^2\bigg)\end{aligned} \tag{16}$$

and

$$\partial\mathcal{L}/\partial\mu_i=\partial\mathcal{L}/\partial\lambda_i=0,\quad i=n_0+1,\cdots,n, \tag{17}$$

where $\partial/\partial\mu_i$ is the ordinary real derivative and $\partial/\partial\lambda_i$ is the complex derivative defined by Brandwood [11]. Here, each partial derivative in (17) is a joint function of μ_i and λ_i. From $\frac{\partial\mathcal{L}}{\partial\mu_i}=0$, optimal μ_i is given in terms of λ_i by

$$\mu_i=\frac{|1+ab\lambda_i|^2-2n\sigma^2(\alpha+\beta a^2|\lambda_i|^2)(1+b^2|\lambda_i|^2)}{2n(\alpha+a^2\beta\lambda_i^2)|1+ab\lambda_i|^2}. \tag{18}$$

By substituting (18) into $\mathcal{L}$, taking the complex derivative of $\mathcal{L}$ w.r.t. λ_i, and performing some manipulation, $\frac{\partial\mathcal{L}}{\partial\lambda_i}=0$ is expressed as *a system of two bivariate polynomial equations with degree seven*:

$$\sum_{k=0}^7\sum_{l_k=0}^k c_{l_k}^{(k)}x_i^{k-l_k}y_i^{l_k}=0\quad\text{and}\quad\sum_{k=0}^7\sum_{l_k=0}^k d_{l_k}^{(k)}x_i^{k-l_k}y_i^{l_k}=0, \tag{19}$$

where x_i and y_i are the real and imaginary parts of λ_i, respectively, i.e., $\lambda_i=x_i+\iota y_i$, and $c_{l_k}^{(k)}$ and $d_{l_k}^{(k)}$ are independent of the bin index i. (The two equations in (19) are from the real and imaginary parts of $\partial\mathcal{L}/\partial\lambda_i=0$.) Here, we have two variables (x_i,y_i) and two nonidentical bivariate polynomial equations. By Bezout's theorem [12], the maximum number of solutions to (19) is the product of the degrees of the two polynomials. Thus, in our case the maximum is $49=7\times 7$, and optimal $\lambda_i=x_i+\iota y_i$ satisfying the KKT condition is one of the solutions $\{\bar{\lambda}_1,\cdots,\bar{\lambda}_{49}\}$ to (19), regardless of i. (If the number of solutions is less than 49, then some of $\bar{\lambda}_j$ are the same.) Due to this fact, the computation of $\bar{R}_{LTI}^{(n)}(P,\gamma P)$ in (15) requires only a finite number of modes. Let n_j, $j=1,\cdots,49$, be the number of occurrence of $\bar{\lambda}_j$ out of $n-n_0$ bins ($n_0+n_1+\cdots+n_{49}=n$). Then, the objective function for maximization in (15) is given by

$$\begin{aligned}\Phi_{LTI}^{(n)} &:= \frac{n_0}{2n}\log\left(1+\frac{\theta_0 n(P-\epsilon_n')}{n_0\sigma^2}\right) \\ &\quad +\frac{1}{2n}\sum_{j=1}^{49}n_j\log\left(1+\frac{\theta_j n(P-\epsilon_n')}{n_j\sigma^2}\cdot\frac{|1+ab\bar{\lambda}_j|^2}{1+b^2|\bar{\lambda}_j|^2}\right)\end{aligned} \tag{20}$$

where θ_j is the portion of the total power allocated to mode j, ($\theta_0+\cdots+\theta_{49}=1$). Based on the above, we now have the capacity for the Gaussian relay channel with LTI relaying, given in the following theorem.

Theorem 2: The capacity of the linear Gaussian relay channel with possibly noncausal LTI relaying is given by

$$\begin{aligned}C_{LTI}(P,\gamma P) &= \max_{\boldsymbol{\tau},\boldsymbol{\theta},\bar{\boldsymbol{\lambda}}}\tau_0\mathcal{C}\left(\frac{\theta_0 P}{\tau_0\sigma^2}\right) \\ &\quad +\sum_{j=1}^{49}\tau_j\mathcal{C}\left(\frac{\theta_j}{\tau_j}\cdot\frac{P}{\sigma^2}\cdot\frac{|1+ab\bar{\lambda}_j|^2}{1+b^2|\bar{\lambda}_j|^2}\right)\end{aligned} \tag{21}$$

subject to $\tau_j,\theta_j\ge 0$, the mode combination constraint $\sum_{j=0}^{49}\tau_j=1$, the power distribution constraint $\sum_{j=0}^{49}\theta_j=1$, and the relay power constraint $\sum_{j=1}^{49}\tau_j|\bar{\lambda}_i|^2\left(a^2\theta_j P/\tau_j+\sigma^2\right)=\gamma P$. Here, $\boldsymbol{\tau}=[\tau_0,\tau_1,\cdots,\tau_{49}]\in\mathbb{R}^{50}$, $\boldsymbol{\theta}=[\theta_0,\theta_1,\cdots,\theta_{49}]\in\mathbb{R}^{50}$, $\bar{\boldsymbol{\lambda}}=[\bar{\lambda}_1,\bar{\lambda}_2,\cdots,\bar{\lambda}_{49}]\in\mathbb{C}^{49}$, and $\mathcal{C}(x)=\frac{1}{2}\log(1+x)$.

Proof: Substitute (20) into (15), and take limit as $n \to \infty$. Then, we have $\epsilon_n, \epsilon'_n, \epsilon''_n \to 0$, $\lim_{n\to\infty} \frac{n_j}{n} = \tau_j$, and the limit of (15) is (21). *(Converse)* The achievable rate cannot be larger than (21) because the maximum number of modes except mode 0 is 49 by Bezout's theorem. *(Achievability)* Suppose that we have obtained $\{\tau_j, \theta_j, \bar{\lambda}_j\}$ from the optimization (21). Shortly, we will see that the above rate can be obtained by partitioning the overall frequency band into 50 subbands and by using IAF with gain $\bar{\lambda}_j$ at subband j. This can be accomplished by using a filter bank of 50 ideal band-pass filters (one for each subband and gain $\bar{\lambda}_j$ for subband j). The impulse response of this filter bank is the sum of the inverse DTFTs of the frequency responses of the subband filters, and is stable. (Causal achievability of the rate (21) will be discussed shortly.) ■

Remark 1: (i) When the number of solutions to (19) is less than 49, (21) is still valid. Solving (21) will yield the same result as solving a possible further-reduced optimization problem in this case. This is like that solving the size n problem (15) directly should yield the same result as solving the reduced-size problem with the cost (20) when the number of solutions is exactly 49. (21) has already finite-letter characterization, but the number of the required modes can be reduced further by considering the structure of the optimization (21). See Corollary 1.

(ii) Since the bins here are frequency bins, a mode can be interpreted as a frequency subband. Later, we will see that the time segment interpretation is also possible.

Corollary 1: The capacity of the linear Gaussian relay channel with possibly noncausal LTI relaying is given by $C_{LTI}(P, \gamma P) =$

$$\max_{\boldsymbol{\tau},\boldsymbol{\theta},\bar{\boldsymbol{\lambda}}} \tau_0 \mathcal{C}\left(\frac{\theta_0 P}{\tau_0 \sigma^2}\right) + \sum_{j=1}^{7} \tau_j \mathcal{C}\left(\frac{\theta_j}{\tau_j} \cdot \frac{P}{\sigma^2} \cdot \frac{(1+ab\bar{\lambda}_j)^2}{1+b^2\bar{\lambda}_j^2}\right) \quad (22)$$

for real a and b, subject to $\tau_j, \theta_j \geq 0$, $\sum_{j=0}^{7} \tau_j = 1$, $\sum_{j=0}^{7} \theta_j = 1$, and $\sum_{j=1}^{7} \tau_j \bar{\lambda}_j^2 \left(a^2\theta_j P/\tau_j + \sigma^2\right) = \gamma P$. Here, $\boldsymbol{\tau} = [\tau_0, \tau_1, \cdots, \tau_7] \in \mathbb{R}^8$, $\boldsymbol{\theta} = [\theta_0, \theta_1, \cdots, \theta_7] \in \mathbb{R}^8$, $\bar{\boldsymbol{\lambda}} = [\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_7] \in \mathbb{R}^7$, and $\mathcal{C}(x) = \frac{1}{2}\log(1+x)$.

Proof: To maximize the argument, $|1+ab\bar{\lambda}_j|^2/(1+b^2|\bar{\lambda}_j|^2)$ in $\mathcal{C}(\cdot)$ in (21), $\bar{\lambda}_j$ should be aligned with the complex conjugate of ab under the same magnitude. Hence, optimal λ_i is real, and we can perform the optimization only over real λ_i without loss of optimality. The same procedure as before can be performed except that $\{\lambda_i\}$ are now real and that $\partial\mathcal{L}/\partial\lambda_i$ is the ordinary real derivative. In this case, λ_i is a solution of a fixed 7th order univariate polynomial equation, $\sum_{k=0}^{7} c_k x^k = 0$ ($c_7 \neq 0$), regardless of i. So, we only need at most seven real $\bar{\lambda}_j$'s. (In the case that a and b are complex, still the phase of optimal $\bar{\lambda}_j$ is fixed and only the magnitude is a single real variable. Thus, we have the same result of at most seven different solutions.) ■

Note that the degree of freedom in real λ_i is halved compared with the complex λ_i case, and the maximum number of solutions to the corresponding KKT conditions is the square-root of that in the complex λ_i case.

Since causal LTI filters are contained in the set of the considered possibly noncausal filters, (21) is an upper bound on the capacity of the Gaussian relay channel with a (single) causal LTI relay filter. To achieve (22) by using a single LTI filter together with a stationary input process, we need a bank of ideal bandpass filters which can be viewed as a single LTI filter as whole and its impulse response is *noncausal*. However, the following theorem shows that still the rate (22) can be achieved by causal linear relay operation by using multiple causal filters over different time segments.

Theorem 3 (Causal achievability): The rate (22) can be achieved by causal linear relay operation by using multiple IAFs over different time segments.

Proof: The causal achievability of (22) can be shown by extending the idea of bursty AF in [3]. Suppose that we are given a solution $(\boldsymbol{\tau}^*, \boldsymbol{\theta}^*, \bar{\boldsymbol{\lambda}}^*)$ to the optimization (22). Then, consider the time interval $0 \leq t \leq T$. Divide the overall time interval into eight time segments with time fraction τ_j for the time segment j. For time segment 0, the source transmits with power $P_0^* = \theta_0^* P/\tau_0^*$ and the relay is turned off. For time segment $j \in \{1, 2, \cdots, 7\}$, the source transmits with power $P_j^* = \theta_j^* P/\tau_j^*$ and the relay operates as IAF with gain $\bar{\lambda}_j^*$. Make T sufficiently large so that the smallest time segment j_{min} is large enough to achieve the corresponding rate $\mathcal{C}\left(\frac{P_{j_{min}}^*}{\sigma^2} \cdot \frac{(1+ab\bar{\lambda}_{j_{min}}^*)^2}{1+b^2\bar{\lambda}_{j_{min}}^{*2}}\right)$. Then, the rate of this causal linear scheme is exactly given by $C_{LTI}(P, \gamma P)$ in (22), and all the source and relay power constraints are satisfied. ■

One could consider more segmentation in time for better performance. In this case, the rate is given by $R_T^{(m)} = \tau_0 \mathcal{C}\left(\frac{P_0}{\sigma_2}\right) + \sum_{j=1}^{m} \tau_j \mathcal{C}\left(\frac{P_j}{\sigma^2} \cdot \frac{(1+ab\bar{\lambda}_j)^2}{1+b^2\bar{\lambda}_j^2}\right)$ for the segmentation of m segments. By the same argument as in the frequency-domain analysis, $m > 7$ does not yield better performance and $m = 7$ is enough. The time segmentation method (nonstationary in time) in Theorem 3 can be considered as the dual approach in time domain of the frequency segmentation method (nonstationary in frequency) with a single LTI filter.

In [3], El Gamal et al. obtained the capacity formula for the frequency-division (FD) linear Gaussian relay channel, given by

$$\begin{aligned} C^{FD-L}(P, \gamma P) \quad = \quad & \max_{\boldsymbol{\tau}^{fd},\boldsymbol{\theta}^{fd},\boldsymbol{\eta}} \tau_0^{fd} \mathcal{C}\left(\frac{\theta_0^{fd} P}{\tau_0^{fd}\sigma^2}\right) \\ & + \sum_{j=1}^{4} \tau_j^{fd} \mathcal{C}\left(\frac{\theta_j^{fd}}{\tau_j^{fd}} \frac{P}{\sigma^2}\left(1 + \frac{a^2b^2\eta_j}{1+b^2\eta_j}\right)\right), \end{aligned} \quad (23)$$

where $\boldsymbol{\tau}^{fd} = [\tau_0^{fd}, \cdots, \tau_4^{fd}]$, $\boldsymbol{\theta}^{fd} = [\theta_0^{fd}, \cdots, \theta_4^{fd}]$, $\boldsymbol{\eta} = [\eta_1, \cdots, \eta_4]$, subject to $\tau_j^{fd}, \theta_j^{fd}, \eta_j \geq 0$, $\sum_{j=0}^{4} \tau_j^{fd} = \sum_{j=0}^{4} \theta_j^{fd} = 1$, and $\sum_{j=1}^{4} \tau_j^{fd}\eta_j \left(a^2\theta_j^{fd} P/\tau_j^{fd} + \sigma^2\right) = \gamma P$. One simple difference of the LTI relay from the FD relay is the maximum number of modes (or segments) required to achieve the capacity. A more important difference lies in the difference in the operation at each mode. In the LTI relay case, the effective signal-to-noise ratio (SNR) at segment j in (22) is given by

$$\frac{P_j}{\sigma^2} \cdot \frac{(1 + ab\bar{\lambda}_j)^2}{1 + b^2\bar{\lambda}_j^2}. \tag{24}$$

This is exactly the effective SNR of the relay channel equipped with IAF with gain $\bar{\lambda}_j$. Thus, Corollary 1 (or Theorem 3) states that *a capacity-achieving strategy is to divide the overall frequency band (or the overall time interval) into at most eight segments and to make the relay behave as an IAF relay with gain $\bar{\lambda}_j$ at segment j*. In the FD relay, on the other hand, the effective SNR in $\mathcal{C}(\cdot)$ in (23) is given by

$$\frac{P_j}{\sigma^2}\left(1 + \frac{a^2b^2\eta_j}{1 + b^2\eta_j}\right) \tag{25}$$

for segment j. Here, let us consider the following data model:

$$\begin{bmatrix} y_{d,1} \\ y_{d,2} \end{bmatrix} = \begin{bmatrix} ab\bar{\lambda}_j \\ 1 \end{bmatrix} x_s + \begin{bmatrix} b\bar{\lambda}_j w_r + w_{d,1} \\ w_{d,2} \end{bmatrix}, \tag{26}$$

where $x_s \sim \mathcal{N}(0, P_j)$ and $w_{d,1}, w_{d,2}, w_r \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$. Note that the data model (26) corresponds to the FD relay channel in which the relay is IAF with gain $\bar{\lambda}_j$. The SNR after optimal matched filtering for the received signal in (26) is given by

$$\frac{P_j}{\sigma^2}\left(1 + \frac{a^2b^2\bar{\lambda}_j^2}{1 + b^2\bar{\lambda}_j^2}\right), \tag{27}$$

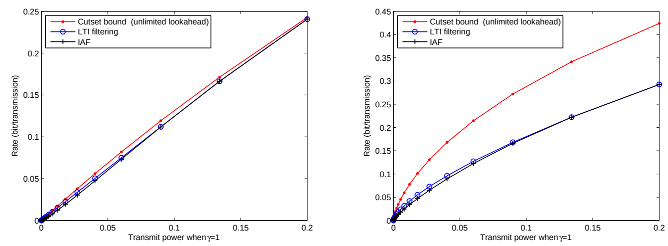
which is exactly the same as (25) with substitution $\eta_j = \bar{\lambda}_j^2$. Hence, (23) states that a capacity-achieving strategy in the linear FD relay is to divide the overall frequency band or time interval into at most five segments and to use IAF at each segment. In both cases, *an optimal strategy achieving the corresponding capacity is to divide the overall frequency band or time interval into a finite number of segments and to use IAF at each segment!* Surprisingly, infinite frequency or time segmentation is not required. The optimality of this finite segmentation comes from the fact that the channel is *flat-fading* and thus each term in the Lagrangian $\mathcal{L}$ in (16) has the same form. In the ISI channel case, the frequency-domain channel coefficients a and b depend on the bin index i. (We should use a_i and b_i instead of a and b.) Hence, the solution (μ_i, λ_i) to $\partial\mathcal{L}/\partial\mu_i = 0$ and $\partial\mathcal{L}/\partial\lambda_i = 0$ can be different for all $i \in \{1, \cdots, n\}$. Thus, in the ISI case, the optimality of finite frequency (or time) segmentation is not guaranteed any more, and the capacity has infinite-letter characterization.

IV. Numerical Results

Eq. (22) was evaluated by using MATLAB. ((21) and (22) resulted in the same value.) Fig. 2 show the rates of several schemes. Since the performance of other schemes is available in [5], we only considered the unlimited look-ahead cut-set bound, IAF and LTI relaying. Fig. 2 (a) shows the performance in the case of $a = 1$, $b = 2$ and $\gamma = 1$. In this case, it is known that the IAF already performs well and achieves the capacity when $P \geq 1/3$ [4]. The LTI relaying improves the performance over the IAF at the very low SNR values, but the gain is not significant. Fig. 2 (b) shows the performance in the case of $a = 2$, $b = 1$ and $\gamma = 1$ in which the IAF has considerable performance degradation from the cut-set bound. Even in this case, the gain by general LTI filtering over the IAF is not so significant. Thus, IAF seems quite sufficient for general single-input single-output (SISO) *flat-fading*[1] Gaussian relay channels when linear filtering is considered for the relay function.

V. Conclusion

We have considered the LTI Gaussian relay channel. By using the Toeplitz distribution theorem and the technique in [3], we have obtained the capacity for LTI relaying in finite-letter characterization, and have shown that the capacity can be achieved by dividing the overall frequency band (or time interval alternatively for causal achievability) into at most eight segments and by using IAF with possibly different gain in each segment. We have provided some numerical results, and the numerical results show that the gain by general LTI filtering over the IAF is not so significant for flat-fading Gaussian relay channels.



(a) (b)

Fig. 2. The performance of LTI relaying: (a) $a = 1, b = 2$ and (b) $a = 2, b = 1$

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