A Study on the Series Expansion of Gaussian Quadratic Forms

Technical Report
WISRL-2012-APR-1
KAIST

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This report is a supplementary document to the paper"Outage Probability and Outage-Based Robust Beamforming for MIMO Interference Channels with Imperfect Channel State Information," by J. Park, Y. Sung, D. Kim and H. V. Poor [Parket12TWC], submitted to *IEEE Transactions on Wireless Communications*.

I. Distribution of a Non-Central Gaussian Quadratic Form

A. Previous work and literature survey

There exist extensive literature about the probability distribution and statistical properties of a quadratic form of non-central (complex) Gaussian random variables in the communications area and the probability and statistics community. Through a literature survey, we found that the main technique to compute the distribution of a central (or a non-central) Gaussian quadratic form is based on series fitting, which was concretely unified and developed by S. Kotz [Kotz-67a, Kotz-67b], and most of other works are its variants, e.g., [Nabar-05]. First, we briefly explain this series fitting method here.

Consider a Gaussian quadratic form $\mathbf{x}^H\bar{\mathbf{Q}}\mathbf{x}$, where $\mathbf{x} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with size n and $\bar{\mathbf{Q}} = \bar{\mathbf{Q}}^H$. The first step of the series fitting method is to convert the non-central Gaussian quadratic form into a linear combination of chi-square random variables:

$$\mathbf{x}^H\bar{\mathbf{Q}}\mathbf{x} = \sum_{i=1}^{n} \lambda_i |z_i + \delta_i|^2 = \sum_{i=1}^{n} \lambda_i [\mathrm{Re}(z_i + \delta_i)^2 + \mathrm{Im}(z_i + \delta_i)^2], \tag{1}$$

where $z_i \overset{independent}{\sim} \mathcal{CN}(0, 2)$ for $i = 1, \cdots, n$, and $\{\delta_i, \lambda_i\}$ are constants determined by $\bar{\mathbf{Q}}$, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Note that $\mathrm{Re}(z_i) \sim \mathcal{N}(0, 1)$ and $\mathrm{Re}(z_i) \sim \mathcal{N}(0, 1)$. Thus, the non-central Gaussian quadratic form is equivalent to a weighted sum of non-central Chi-square random variables of which moment generating function (MGF) is *known*. The MGF of a weighted sum of n independent non-central χ^2 random variables with degrees of freedom $2m_i$ and non-centrality parameter μ_i^2 is given by

$$\Phi(s) = \exp\{-\frac{1}{2}\sum_{i=1}^{n} \mu_i^2 + \frac{1}{2}\sum_{i=1}^{n} \frac{\mu_i^2}{1 - 2\lambda_i s}\} \cdot \prod_{i=1}^{n} \frac{1}{(1 - 2\lambda_i s)^{m_i}}. \tag{2}$$

Note here that $\Phi(-s)$ is nothing but the Laplace transform of the *probability density function (PDF)* of $\mathbf{x}^H\bar{\mathbf{Q}}\mathbf{x}$ or equivalently $\sum_{i=1}^{n} \lambda_i |z_i + \delta_i|^2$. Now, the series fitting method expresses the PDF as an infinite series composed of a set of known basis functions and tries to find the linear combination coefficients so that the Laplace transform of this series is the same as the known

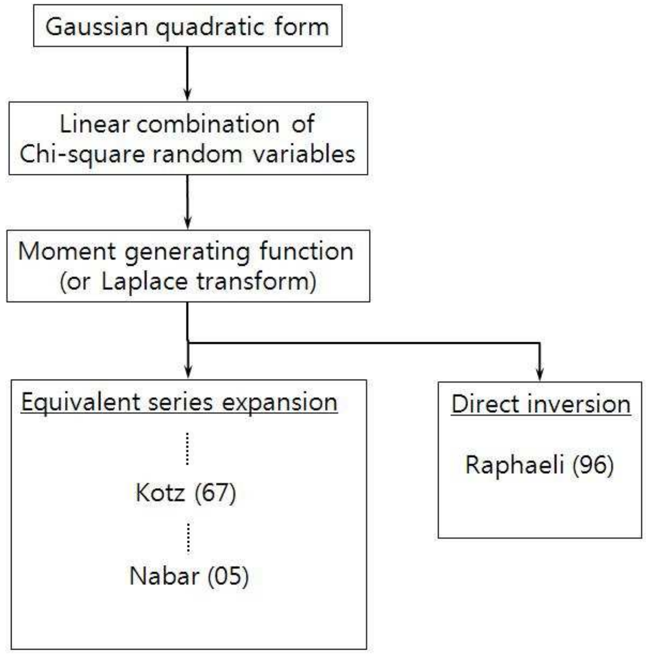


Fig. 1. Computation of the distribution of a Gaussian quadratic form

$\Phi(-s)$. Specifically, let the PDF be

$$g_n(\bar{\mathbf{Q}}, \boldsymbol{\mu}, \boldsymbol{\Sigma}; y) = \sum_{k=0}^{\infty} c_k h_k(y), \tag{3}$$

where $\{h_k(y), k = 0, 1, \cdots\}$ is the set of known basis functions and $\{c_k, k = 0, 1, \cdots\}$ is the set of linear combination coefficients to be determined. Here, to make the problem tractable, in most cases, the following conditions are imposed. First, the sequence $\{h_k(y)\}$ of basis functions is chosen among measurable complex-valued functions on $[0, \infty]$ such that

$$\sum_{k=0}^{\infty} |c_k||h_k(y)| \leq Ae^{by}, \qquad y \in [0, \infty] \text{ almost everywhere}, \tag{4}$$

where A and b are real constants. Second, the Laplace transform $\hat{h}_k(s)$ of $h_k(y)$ has a special form:

$$\hat{h}_k(s) = \xi(s)\eta^k(s), \tag{5}$$

where $\xi(s)$ is a non-vanishing, analytic function for $\text{Re}(s) > b$, and $\eta(s)$ is analytic for $\text{Re}(s) > b$ and has an inverse function. The first condition is for the existence of Laplace transform and the second condition is to make the problem tractable. Finally, with the pre-determined $\{h_k(y)\}$

with the conditions, the coefficients $\{c_k\}$ are computed so that

$$\mathcal{L}(g_n(\bar{\mathbf{Q}}, \boldsymbol{\mu}, \boldsymbol{\Sigma}; y)) = \sum_{k=0}^{\infty} c_k \hat{h}_k(s) = \Phi(-s), \tag{6}$$

where $\mathcal{L}(\cdot)$ denote the Laplace transform of a function.

Widely used $\{h_k(y)\}$ for the series expansion of the PDF of a quadratic form of non-central Gaussian random variables is as follows. [Kotz-67a, Kotz-67b]

1. (Power series): $h_k(y) = (-1)^k \frac{(y/2)^{n/2+k-1}}{2\Gamma(n/2+k)}$.
2. (Laguerre polynomials):

$$h_k(y) = g(n; y/\beta)[k! \frac{\Gamma(n/2)}{\beta\Gamma(n/2+k)}] L_k^{(n/2-1)}(y/2\beta), \tag{7}$$

where $g(n; y)$ is the central χ^2 density with n degrees of freedom and $L_k^{(n/2-1)}(x)$ is the generalized Laguerre polynomial defined by Rodriges' formula

$$L_k^{(n/2-1)}(x) = \frac{1}{k!} e^x x^{-(n/2-1)} \frac{d^k}{dx^k} e^{-x} x^{k+1}$$

for $a > 1$ and a positive control parameter β.

For the detail computation of $\{c_k\}$, please refer to [Kotz-67a, Kotz-67b, Mathai-92]. The whole procedure is summarized in Fig. 1.

Reference group 1

[Kotz-67a] S. Kotz, N. L. Johnson, and D. W. Boyd, "Series representation of distributions of quadratic forms in normal variables. I. Central Case," *Ann. Math. Statist.,* vol, 38, pp. 823 – 837, Jun. 1967.

[Kotz-67b] S. Kotz, N. L. Johnson, and D. W. Boyd, "Series representation of distributions of quadratic forms in normal variables. II. Non-central Case," *Ann. Math. Statist.,* vol. 38, pp. 838 – 848, Jun. 1967.

[Mathai-92] A. M. Mathai and S. B. Provost, *Quadratic forms in random variables: Theory and applications*, New York:M. Dekker, 1992.

[Nabar-05] R. Nabar, H. Bolcskei, and A. Paulraj, "Diversity and Outage Performance of Space-Time Block Coded Ricean MIMO Channels", *IEEE Trans. on Wireless Commun.*, vol. 4, no. 5, Sept. 2005.

Reference group 2

[Pachares-55] J. Pachares, “Note on the distribution of a definite quadratic form,” *Ann. Math. Statist.,* vol. 26, pp. 128 – 131, Mar. 1955. ⇒ Power series representation of quadratic form of central Gaussian random variables.

[Shah-61] B. K. Shah and C. G. Khatri, “Distribution of a definite quadratic form for non-central normal variates,” *Ann. Math. Statist.,* vol. 32, pp. 883 – 887, Sep. 1961. ⇒ Power series representation of quadratic form of non-central Gaussian random variables.

[Shah-63] B. K. Shah, “Distribution of definite and of indefinite quadratic forms from a non-central normal distribution,” *Ann. Math. Statis.,* vol. 34, pp. 186 – 190, Mar. 1963. ⇒ Extends [Gurland-55] to derive a representation of quadratic form of non-central Gaussian random vector with Laguerre polynomial. Double series of Laguerre polynomials is required.

[Gurland-55] J. Gurland, “Distribution of definite and indefinite quadratic forms,” *Ann. Math. Statist.,* vol. 26, pp. 122 – 127, Jan. 1955. ⇒ Provides a simple representation of quadratic form of central Gaussian random vector in Laguerre polynomial.

[Gurland-56] J. Gurland, “Quadratic forms in normally distributed random variables,” *Sankhya: The Indian Journal of Statistics* vol. 17, pp. 37 – 50, Jan. 1956. ⇒ CDF for the indefinite quadratic form of central random variable.

[Ruben-63] H. Ruben, “A new result on the distribution of quadratic forms,” *Ann. Math. Statist.,* vol. 34, pp. 1582 – 1584, Dec. 1963. ⇒ Represents the CDF of quadratic form of central and non-central Gaussian random vector with central/non-central χ^2 distribution function.

[Tiku-65] M. L. Tiku, “Laguerre series forms of non central χ^2 and F distributions,” *Biometrika,* vol. 52, pp. 415 – 427, Dec. 1965. ⇒ Another series representaion with Laguerre polynomials.

[Davis-77] A. W. Davis, “A differential equation approach to linear combinations of independent chi-squares,” *J. of the Ame. Statist. Assoc.* vol. 72, pp. 212 – 214, Mar. 1977. ⇒ Provides another series representation with power series.

[Imhof-61] J. P. Imhof, “Computing the distribution of quadratic forms in normal variables,” *Biometrika* vol. 48, pp. 419 – 426, Dec. 1961. ⇒ Provides a numerical method of computing the distribution

[Rice-80] S. O. Rice, “Distribution of quadratic forms in normal variables - Evaluation by numerical integration,” *SIAM J. Scient. Statist. Comput.,* vol. 1, no. 4, pp. 438 – 448, 1980. ⇒ Another numerical method of computing distribution.

[Biyari-93] K. H. Biyari and W. C. Lindsey, “Statistical distribution of Hermitian quadratic forms in complex Gaussian variables," *IEEE Trans. Inform. Theory,* vol. 39, pp. 1076 – 1082, Mar. 1993. ⇒ Series expansion of multi-variate complex Gaussian random variables. This paper deals with the case that the Hermitian matrix in the quadratic form is a special block-diagonal matrix.

Reference group 3

[Raphaeli-96] D. Raphaeli, "Distribution of noncentral indefinite quadratic forms in complex normal variables," *IEEE Trans. Inf. Theory,* vol. 42, pp. 1002 – 1007, May 1996.

[Al-Naffouri-09] T. Al-Naffouri and B. Hassibi, "On the distribution of indefinite quadratic forms in Gaussian random variables," in *Proc. of IEEE Int. Symp. Inform. Theory*, (Seoul, Korea), Jun.–Jul. 2009.

B. The difference of our work from the previous works

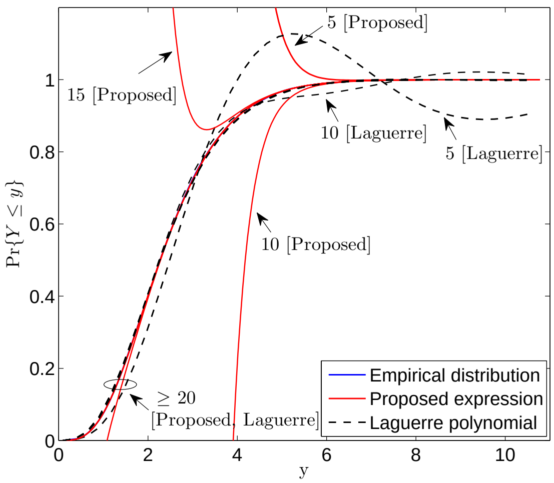
First, let us remind our outage event in MIMO interference channels. From equations (5), (6) and (7) in [Parket12TWC], we have

$$\Pr\{\text{outage}\} = \Pr\left\{ \sum_{i=1}^{K}\sum_{j=1}^{d} X_{ki}^{(mj)H} X_{ki}^{(mj)} \geq \frac{|\mathbf{u}_k^{(m)H}\hat{\mathbf{H}}_{kk}\mathbf{v}_k^{(m)}|^2}{2^{R_k^{(m)}}-1} - \sigma^2 =: \tau \right\}, \tag{8}$$

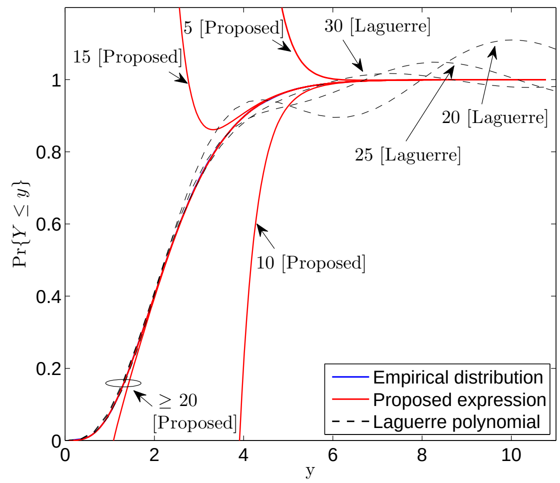
where $X_{ki}^{(mj)}$ is a non zero-mean Gaussian random variable. Note that the outage probability is an *upper* tail probability of the distribution of the Gaussian quadratic form $\sum_{j=1}^{d} X_{ki}^{(mj)H} X_{ki}^{(mj)}$. However, as seen in Fig. 2, *the most widely-used series fitting method explained in the previous subsection yields a good approximation of the distribution at the lower tail not at the upper tail.* The discrepancy between the series and the true PDF is large at the upper* tail for a truncated series. *On the other hand, our approach yields a good approximation to the true distribution at the upper tail.* Thus, the proposed series is more relevant to our problem than the series fitting method.

Our approach to the upper tail approximation is based on the recent works by Raphaeli [Raphaeli-96] and by Al-Naffouri and Hassibi [Al-Naffouri-09]. First, let us explain Raphaeli's method. The procedure in Fig. 1 up to obtaining the MGF of the Gaussian quadratic form is common to both the sequence fitting method and Raphaeli's method. However, Raphaeli's method obtains the PDF by direct inverse Laplace transform of the MGF $\Phi(s)$. Typically, the

*In the case of the problem considered in [Nabar-05], the outage defined in [Nabar-05] is associated with the lower tail of the distribution and thus the series fitting method is well suited to that case. However, our system setup and considered problem are different from those in [Nabar-05].



(a)



(b)

Fig. 2. Series fitting method versus direct inverse Laplace transform method: number of variables = 4, $\boldsymbol{\mu} = 0.5\mathbf{1}$, $\bar{\mathbf{Q}} = [1, 0.5, 0, 0; 0.5, 1, 0, 0; 0, 0, 1, 0; 0, 0, 0, 1]$, and $\boldsymbol{\Sigma} = 0.3\mathbf{I}$. (a) $\beta = 1$ and (b) $\beta = 2$. (β is the control parameter for the Laguerre polynomials in (7).) Note that the convergent speed of the series fitting method based on the Laguerre polynomials depends much on β. In the case of $\beta = 2$, the series fitting method based on the Laguerre polynomials yields large errors at the upper tail. It is not simple how to choose β and an efficient method is not known. (One cannot run simulations for empirical distributions for all cases.) The series fitting method based on the power series shows bad performance, and it cannot be used in practice.

inverse Laplace transform of the MGF is represented as a complex contour integral and then the complex contour integral is computed as an infinite series by the residue theorem. However, to obtain the cumulative distribution function (CDF), which is actually necessary to compute the tail probability, Raphaeli's method requires one more step, the integration of the PDF, to obtain the CDF since the MGF $\Phi(s)$ is the Laplace transform of the *PDF*.

In [Parket12TWC], to obtain the CDF of a general Gaussian quadratic form, we did not use the MGF $\Phi(s)$, which is a bit complicated and requires an additional step, like Raphaeli, but instead we directly used a simple contour integral for the CDF, eq. (12) in [Parket12TWC], obtained by Al-Naffouri and Hassibi [Al-Naffouri-09].† Then, the contour integral was computed as an

†In [Al-Naffouri-09], Al-Naffouri and Hassibi obtained the contour integral, eq. (12) in [Parket12TWC] for the CDF of a Gaussian quadratic form. However, they did not obtain closed-form series expressions for the contour integral in general cases except a few simple cases. The main goal of [Al-Naffouri-09] was to derive a nice and simple contour integral form for the CDF.

infinite series by the residue theorem. (Using the residue theorem is borrowed from Raphaeli's work.) Thus, our result is simpler than Raphaeli's approach and does not require the integration of a PDF for the CDF.

As mentioned already, the series expansion in [Parket12TWC] has a particular advantage over the series fitting method considered in [Nabar-05] for the outage event defined in [Parket12TWC]; The series in [Parket12TWC] fits the upper tail of the distribution well with a few number of terms. We shall provide a detailed proof for this in a special case in the next subsection. Thus, our series expressions for outage probability in MIMO interference channels are meaningful and relevant.

II. Computational Issues and Convergence of the Obtained Series

A. Computing higher order derivatives

Recall the general outage expression in Theorem 1 in [Parket12TWC]:

$$\begin{aligned} \Pr\{\text{outage}\} &= \Pr\{\log_2(1+\mathsf{SINR}_k^{(m)}) \le R_k^{(m)}\} \\ &= -\sum_{i=1}^{\kappa} \frac{e^{-(\frac{\tau}{\lambda_i}+\sum_{j=1}^{\kappa_i}|\chi_i^{(j)}|^2)}}{\lambda_i^{\kappa_i}} \sum_{n=\kappa_i-1}^{\infty} \frac{1}{n!} g_i^{(n)}(0) \frac{1}{(n-\kappa_i+1)!} \left(\frac{\sum_{j=1}^{\kappa_i}|\chi_i^{(j)}|^2}{\lambda_i}\right)^{n-\kappa_i+1} \end{aligned} \tag{9}$$

where

$$g_i(s) = \frac{e^{\tau s}}{s-1/\lambda_i} \cdot \frac{\exp\left(-\sum_{p\neq i} \frac{(s-1/\lambda_i)\lambda_p}{1+(s-1/\lambda_i)\lambda_p} \sum_{q=1}^{\kappa_p} |\chi_p^{(q)}|^2\right)}{\prod_{p\neq i}\left(1+(s-1/\lambda_i)\lambda_p\right)^{\kappa_p}}. \tag{10}$$

To compute (9), we need to compute

- $\{\lambda_i\}$ (the eigenvalues of the $Kd \times Kd$ covariance matrix $\mathbf{\Sigma} = \mathbf{\Psi}\mathbf{\Lambda}\mathbf{\Psi}^H$),
- $\{\chi_i^{(j)}\}$ (the elements of Kd vector $\boldsymbol{\chi} = \mathbf{\Lambda}^{-1/2}\mathbf{\Psi}^H\boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is the mean vector of the Gaussian distribution),
- and the higher order derivatives of $g_i(s)$.

The computation of $\{\lambda_i\}$ and $\{\chi_i^{(j)}\}$ is simple since the sizes of the mean vector and the covariance matrix are Kd and $Kd \times Kd$, respectively. Furthermore, the higher order derivatives of $g_i(s)$ can also be computed efficiently based on recursion [Mathai-92],[Raphaeli-96]. Note that $g_i(s) =$

$e^{\log g_i(s)}$. Thus, the derivative of $g_i(s)$ can be written as

$$\begin{aligned} g_i^{(1)}(s) &= g_i(s)[\log g_i(s)]^{(1)}, \\ g_i^{(2)}(s) &= g_i^{(1)}(s)[\log g_i(s)]^{(1)} + g_i(s)[\log g_i(s)]^{(2)}, \\ &\vdots \\ g_i^{(n)}(s) &= \sum_{l=0}^{n-1} \binom{n-1}{l} g_i^{(l)}(s)[\log g_i(s)]^{(n-l)}, \qquad n \geq 1 \end{aligned} \tag{11}$$

where $g_i^{(l)}(s)$ and $[\log g_i(s)]^{(l)}$ denote the l-th derivatives of $g_i(s)$ and $\log g_i(s)$, respectively. Here, $[\log g_i(s)]^{(n)}$ can be computed from (10) as

$$[\log g_i(s)]^{(n)} = \tau\delta_{1n} - \frac{(n-1)!(-1)^{n-1}}{(s-1/\lambda_i)^n} - \sum_{p\neq i} \frac{n!(-1)^{n-1}\lambda_p^n}{(1+\lambda_p(s-1/\lambda_i))^{n+1}} \sum_{q=1}^{\kappa_p} |\chi_p^{(q)}|^2 - \sum_{p\neq i} \frac{(n-1)!(-1)^{n-1}\kappa_p\lambda_p^n}{(1+\lambda_p(s-1/\lambda_i))^n}$$

where δ_{1n} is Kronecker delta function. Thus, for given $g_i(s)$ and $[\log g_i(s)]^{(l)}$, we can compute $g_i^{(l)}(s)$ efficiently in a recursive way, as shown in (11).

B. Convergence analysis

In this subsection, we provide some convergence analysis on the derived series expansion in [Parket12TWC]. Consider the general result in Theorem 1 of [Parket12TWC] for the CDF of a Gaussian quadratic form:

$$\Pr\{Y \leq y\} = 1 + \sum_{i=1}^{\kappa} \frac{e^{-(\frac{y}{\lambda_i} + \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)}}{\lambda_i^{\kappa_i}} \sum_{n=\kappa_i-1}^{\infty} \frac{1}{n!} g_i^{(n)}(0,y) \frac{1}{(n-\kappa_i+1)!} \left(\frac{\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}{\lambda_i} \right)^{n-\kappa_i+1} \tag{12}$$

where

$$g_i(s,y) = \frac{e^{sy}}{s-\lambda_i^{-1}} \cdot \frac{\exp\left(-\sum_{p\neq i} \frac{(s-1/\lambda_i)\lambda_p}{1+(s-1/\lambda_i)\lambda_p} \sum_{q=1}^{\kappa_p} |\chi_p^{(q)}|^2\right)}{\prod_{p\neq i}\left(1+(s-1/\lambda_i)\lambda_p\right)^{\kappa_p}}.$$

Here, we explicitly use the variable y as an input parameter of the function $g_i(s)$ for later explanation. $g_i^{(n)}(s,y)$ denotes the n-th partial derivative of $g_i(s,y)$ with respect to s. (Here, κ is the number of distinct eigenvalues of the $Kd \times Kd$ covariance matrix $\mathbf{\Sigma}$ and κ_i is the geometric order of eigenvalue λ_i. $\sum_{i=1}^{\kappa} \kappa_i = Kd$.) The residual error caused by truncating the infinite series after the first N terms is given by

$$R_N(y) = \sum_{i=1}^{\kappa} \frac{e^{-(\frac{y}{\lambda_i} + \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)}}{\lambda_i^{\kappa_i}} \sum_{n=N+1}^{\infty} \frac{1}{n!} g_i^{(n)}(0,y) \frac{1}{(n-\kappa_i+1)!} \left(\frac{\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}{\lambda_i} \right)^{n-\kappa_i+1}, \tag{13}$$

and we have

$$\Pr\{Y \leq y; \text{infinite sum}\} = \Pr\{Y \leq y; \text{truncation at } N\} + R_N(y).$$

The truncation error $R_N(y)$ can be expressed as

$$R_N(y) = \sum_{i=1}^{\kappa} R_N^i(y), \tag{14}$$

where

$$R_N^i(y) = \frac{e^{-(\frac{y}{\lambda_i} + \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2)}}{\lambda_i^{\kappa_i}} \sum_{n=N+1}^{\infty} \frac{1}{n!} g_i^{(n)}(0,y) \frac{1}{(n-\kappa_i+1)!} \left(\frac{\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}{\lambda_i} \right)^{n-\kappa_i+1} \tag{15}$$

for each $1 \leq i \leq \kappa$. Then, the magnitude of each term $|R_N^i(y)|$ in the truncation error is bounded as

$$|R_N^i(y)| \leq \frac{1}{\lambda_i^{\kappa_i}} \exp\left\{ -\left(\frac{y}{\lambda_i} + \sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2 \right) \right\} \cdot \sum_{n=N+1}^{\infty} \frac{1}{n!} \left| g_i^{(n)}(0,y) \right| \cdot \frac{1}{(n-\kappa_i+1)!} \left(\frac{\sum_{j=1}^{\kappa_i} |\chi_i^{(j)}|^2}{\lambda_i} \right)^{n-\kappa_i+1}. \tag{16}$$

As seen in Fig. 2, our series expansion fits the upper tail distribution first. Now, to assess the overall convergence speed of our series, for the same step as in Fig. 2, we ran some simulations to obtain an empirical distribution, and computed the overall mean square error (MSE) between the truncated series and the empirical distribution over $0 \leq y \leq 10$ as

$$\text{CDF MSE} = \frac{1}{200} \sum_{i=1}^{200} \Big| \Pr\{Y \leq y_i; N, \text{type of series}\} - \Pr\{Y \leq y_i; \text{empirical}\} \Big|^2,$$

where $\{y_i\}$ are the uniform samples of $[0, 10]$. Fig. 3 shows the CDF MSE of the three methods in Fig. 2: the proposed series, the series fitting method with $\beta = 1$ and the series fitting method with $\beta = 2$. It is seen in Fig. 3 that the overall convergence of the proposed series can be worse than the series fitting method at the small values for the number of summation terms for the setting in Fig. 2. The bad overall convergence is due to worse fitting at the lower tail of the distribution, but the bad lower tail approximation is not important to our outage computation. (Please see Fig. 2.) Fig. 4 shows another case. In this case, the proposed series outperforms the series fitting method both in the overall convergence and in the upper tail convergence. It is seen numerically that the proposed series fits the upper tail distribution first. Now, we shall prove this property of the proposed series. However, it is a difficult problem to prove this property in general cases. Thus, in the next subsection, we provide a proof of this property when the number of distinct eigenvalues of the covariance matrix $\mathbf{\Sigma}$ is one, e.g., in the i.i.d. case.

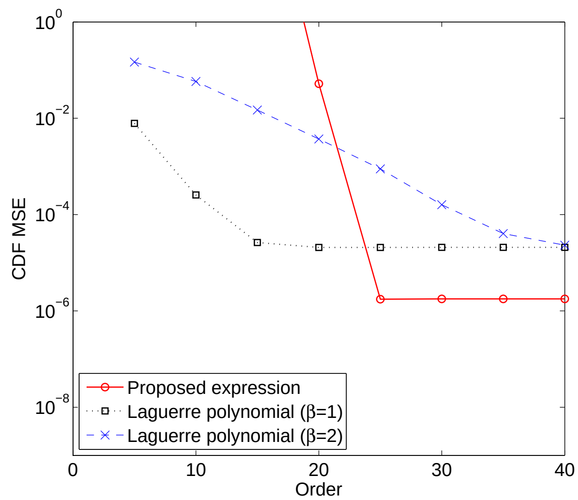
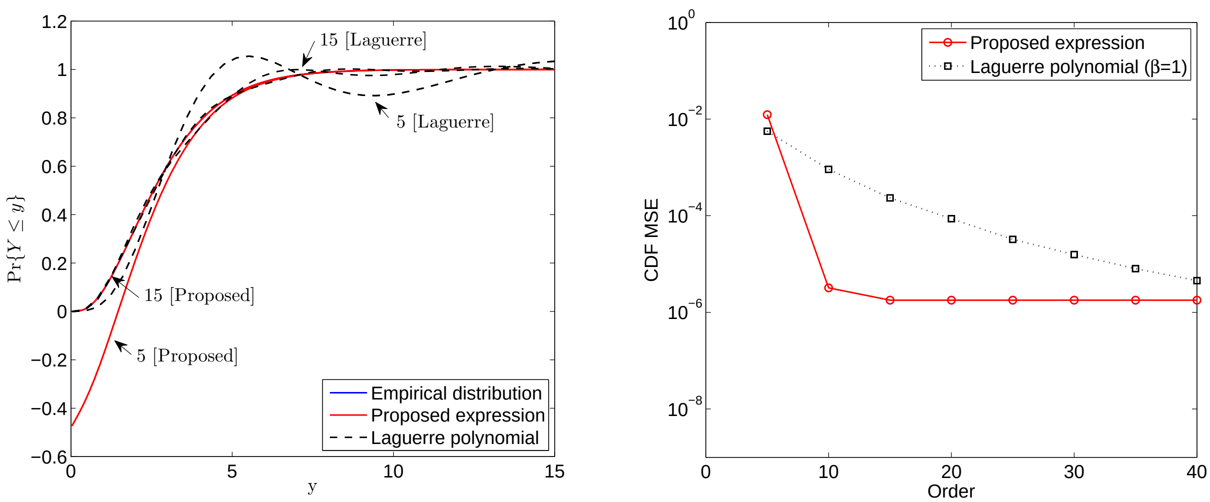


Fig. 3. CDF MSE of the CDFs in Fig. 2



(a) (b)

Fig. 4. number of variables = 4, $\boldsymbol{\mu} = 0.5\mathbf{1}$, $\bar{\mathbf{Q}} = \mathbf{I}$, and $\boldsymbol{\Sigma} = [0.2641\ 0.0328\ 0.1963\ 0.1140;\ 0.0328\ 0.6097\ -0.1739\ 0.1708;\ 0.1963\ -0.1739\ 0.8746\ -0.0022;\ 0.1140\ 0.1708\ -0.0022\ 0.1250]$. In this case eigenvalues are 1.0000, 0.6318, 0.2158, and 0.0259 with $\beta = 1$. (a) CDF, (b) CDF MSE. Uniform sample of y is taken over $[0, 15.9]$.

B.1 The identity covariance matrix case

Suppose that there is only one eigenvalue, λ (> 0), with multiplicity κ for the covariance matrix $\boldsymbol{\Sigma}$. This case corresponds to Corollary 4 in [Parket12TWC], and the outage probability is given by

$$\Pr\{Y \le y\} = 1 + \frac{\exp(-\eta^2)}{\lambda^\kappa} \exp\left(-\frac{y}{\lambda}\right) \sum_{n=\kappa-1}^{\infty} g^{(n)}(0,y) \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!}, \tag{17}$$

where

$$g(s,y) = \frac{e^{ys}}{s-\lambda^{-1}} \tag{18}$$

and $\eta^2 = \sum_{j=1}^{\kappa} |\chi^{(j)}|^2$. The residual error caused by truncating the infinite series after the first N terms is given by

$$R_N(y) = \frac{\exp(-\eta^2)}{\lambda^\kappa} \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} g^{(n)}(0,y) \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!}. \tag{19}$$

Before we proceed, we first obtain the n-th derivative of $g(s,y)$ at $s=0$, which is given in the following lemma.

Lemma 1: For $n \ge 0$,

$$g^{(n)}(0,y) = -\lambda \sum_{k=0}^{n} \frac{n!}{(n-k)!} \lambda^k y^{n-k}. \tag{20}$$

Proof: Proof is given by induction. The validity of the claim for $n = 0, 1$ and 2 is shown by direction computation:

$$\begin{aligned}
g^{(0)}(0,y) &= = \left.\frac{ye^{ys}}{s-1/\lambda}\right|_{s=0} = -\lambda = -\lambda \sum_{k=0}^{0} \frac{0!}{(0-k)!} \lambda^k y^{0-k}, \\
g^{(1)}(0,y) &= = \left.\frac{ye^{ys}(s-1/\lambda) - e^{ys}}{(s-1/\lambda)^2}\right|_{s=0} = -\lambda(y+\lambda) = -\lambda \sum_{k=0}^{1} \frac{1!}{(1-k)!} \lambda^k y^{1-k}, \\
g^{(2)}(0,y) &= \left.\frac{\left(ye^{ys}(ys - y/\lambda - 1) + e^{ys}y\right)(s-\frac{1}{\lambda})^2 - 2e^{ys}(ys - y/\lambda - 1)(s-\frac{1}{\lambda})}{(s-1/\lambda)^4}\right|_{s=0} \\
&= -\lambda(y^2 + 2\lambda y + 2\lambda^2) = -\lambda \sum_{k=0}^{2} \frac{2!}{(2-k)!} \lambda^k y^{2-k}.
\end{aligned}$$

Now, suppose that (20) holds up to the $(n-1)$-th derivative of $g(s,y)$. From the recursive formula in (11), $g^{(n)}(0,y)$ is obtained as

$$\begin{aligned}
&g^{(n)}(0,y) \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} g^{(k)}(0,y) (\log g(0,y))^{(n-k)} \\
&= \binom{n-1}{0} g^{(0)}(0,y)(\log g(0,y))^{(n)} + \binom{n-1}{1} g^{(1)}(0,y)(\log g(0,y))^{(n-1)} + \cdots + \binom{n-1}{n-1} g^{(n-1)}(0,y)(\log g(0,y))^{(1)}.
\end{aligned} \tag{21}$$

Since $[\log g(s)] = ys - \log(s - 1/\lambda)$, we can easily see that $[\log g(0)]^{(1)} = y + \lambda$ and $[\log g(0)]^{(n)} = (n-1)!\lambda^n$ for $n \geq 2$. Therefore, (21) can be rewritten as

$$
\begin{aligned}
g^{(n)}(0,y) =&(n-1)!g(0,y)\lambda^n + (n-1)g^{(1)}(0,y)(n-2)!\lambda^{n-1} + \binom{n-1}{2} g^{(2)}(0,y)(n-3)!\lambda^{n-2} + \cdots \\
&+(n-1)g^{(n-2)}(0,y)\lambda^2 + g^{(n-1)}(0,y)(y+\lambda) \\
=&(n-1)!g(0,y)\lambda^n + (n-1)!g^{(1)}(0,y)\lambda^{n-1} + \frac{(n-1)!}{2} g^{(2)}(0,y)\lambda^{n-2} + \cdots \\
&+ (n-1)g^{(n-2)}(0,y)\lambda^2 + \lambda g^{(n-1)}(0,y) + y g^{(n-1)}(0,y) \\
\overset{(a)}{=}& -\lambda \left[\sum_{l=0}^{n-1} \frac{(n-1)!}{l!} \left(\sum_{k=0}^{l} \frac{l!}{(l-k)!} \lambda^k y^{l-k} \right) \lambda^{n-l} + y \sum_{m=0}^{n-1} \frac{(n-1)!}{(n-m-1)!} \lambda^m y^{n-m-1} \right] \\
=& -\lambda \left[\sum_{l=0}^{n-1} \frac{(n-1)!}{l!} \left(\sum_{k=0}^{l} \frac{l!}{(l-k)!} \lambda^k y^{l-k} \right) \lambda^{n-l} + \sum_{m=0}^{n-1} \frac{(n-1)!}{(n-m-1)!} \lambda^m y^{n-m} \right] \qquad (22)
\end{aligned}
$$

where (a) holds since (20) holds for all $g^{(0)}(0,y), \cdots, g^{(n-1)}(0,y)$ by the induction assumption.

Here, consider the coefficient of each y^i in (22) for $i = 0, \cdots, n$.

i) y^n is obtained only when $m = 0$. The coefficient of y^n from (22) is therefore given by $-\lambda$. It corresponds to the coefficient of y^n in (20).

ii) For $0 < p \leq n$, the coefficient of y^{n-p} is obtained by considering all (l,k) that satisfies $l - k = n - p$ due to the first term in the right-hand side (RHS) of (22), and $m = p$ due to the second term of the RHS of (22). In the first case, we obtain y^{n-p} with the following pairs $(l,k) = (n-1, p-1)$, $(n-2, p-2)$, $\cdots$, $(n-p, 0)$. For these (l,k) pairs, we have

$$
-\lambda \sum_{l=n-p}^{n-1} \frac{(n-1)!}{l!} \cdot \left(\frac{l!}{(n-p)!} \lambda^{l-n+p} y^{n-p} \right) \cdot \lambda^{n-l} = -\lambda \sum_{l=n-p}^{n-1} \frac{(n-1)!}{(n-p)!} \lambda^p y^{n-p} = -\lambda p \frac{(n-1)!}{(n-p)!} \lambda^p y^{n-p}. \qquad (23)
$$

In the second case of $m = p$, we have

$$
-\lambda \frac{(n-1)!}{(n-p-1)!} \lambda^p y^{n-p}. \qquad (24)
$$

Finally, the coefficient of y^{n-q} is given by adding (23) and (24):

$$
\begin{aligned}
& -\lambda \left(\frac{(n-1)!}{(n-p-1)!} + p \frac{(n-1)!}{(n-p)!} \right) \lambda^p y^{n-p} \\
= & -\lambda \frac{(n-1)!}{(n-p-1)!} \left(1 + \frac{p}{n-p} \right) \lambda^p y^{n-p} \\
= & -\lambda \frac{n!}{(n-p)!} \lambda^p y^{n-p},
\end{aligned}
$$

which is equivalent to the coefficient for y^{n-p} in (20) $(0 < p \leq n)$. Thus, (20) holds for $g^{(n)}(0, y)$. ■

Note that $g^{(n)}(0, y) < 0$ for all $n \geq 0$ from (20). Therefore, $R_N(y) \leq 0$ for all N and y and $|g^{(n)}(0, y)| = -g^{(n)}(0, y)$.

Now, consider the residual error term $R_N(y)$ in (19). The magnitude of the residual error can be upper bounded as follows:

$$\begin{aligned}
|R_N(y)| &= \frac{\exp(-\eta^2)}{\lambda^\kappa} \cdot \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} |g^{(n)}(0, y)| \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!} \\
&= \frac{\exp(-\eta^2)}{\lambda^\kappa} \cdot \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} (-g^{(n)}(0, y)) \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!} \\
&= -\frac{\exp(-\mu^2)}{\lambda^\kappa} \cdot \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} g^{(n)}(0, y) \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!} \\
&= -\frac{\exp(-\eta^2)}{\lambda^\kappa} \cdot \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} \frac{1}{n!} g^{(n)}(0, y) \left(\frac{1}{2\lambda}\right)^n \frac{(2\eta^2)^{n-\kappa+1}(2\lambda)^{\kappa-1}}{(n-\kappa+1)!} \\
&= -(2\lambda)^{\kappa-1} \cdot \frac{\exp(-\eta^2)}{\lambda^\kappa} \cdot \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} \frac{1}{n!} g^{(n)}(0, y) \left(\frac{1}{2\lambda}\right)^n \frac{(2\eta^2)^{n-\kappa+1}}{(n-\kappa+1)!} \\
&\overset{(a)}{\leq} -(2\lambda)^{\kappa-1} \cdot \frac{\exp(-\eta^2)}{\lambda^\kappa} \cdot \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} \frac{1}{n!} g^{(n)}(0, y) \left(\frac{1}{2\lambda}\right)^n \exp(2\eta^2) \\
&= -\frac{2^{\kappa-1}}{\lambda} \exp(\eta^2) \cdot \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} \frac{1}{n!} g^{(n)}(0, y) \left(\frac{1}{2\lambda}\right)^n \\
&\overset{(b)}{\leq} -\frac{2^{\kappa-1}}{\lambda} \exp(\eta^2) \cdot \exp\left(-\frac{y}{\lambda}\right) \cdot \sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)}(0, y) \left(\frac{1}{2\lambda}\right)^n \\
&\overset{(c)}{=} -\frac{2^{\kappa-1}}{\lambda} \exp(\eta^2) \cdot \exp\left(-\frac{y}{\lambda}\right) \cdot g\left(\frac{1}{2\lambda}, y\right) \\
&\overset{(d)}{=} -\frac{2^{\kappa-1}}{\lambda} \exp(\eta^2) \cdot \exp\left(-\frac{y}{\lambda}\right) \cdot \frac{\exp(y/2\lambda)}{-1/2\lambda} \\
&= 2^\kappa \exp(\eta^2) \cdot \exp\left(-\frac{y}{2\lambda}\right)
\end{aligned} \tag{25}$$

where (a) is from $\frac{\gamma^k}{k!} \leq \exp(\gamma) = \sum_{p=0}^{\infty} \gamma^p/p!$ for any $\gamma > 0$, (b) is from the fact that summand is negative, (c) is by using the Taylor series expansion, and (d) is from (18). Since η is a fixed constant, from (25), for any $N \geq 0$

$$\lim_{y \to \infty} |R_N(y)| = 0. \tag{26}$$

Thus, it is clear that the proposed series converges from the upper tail distribution!

Now, let us consider the residual error magnitude as a function of y for given N. From (20), we have

$$\frac{\partial g^{(n)}(0,y)}{\partial y} = n g^{(n-1)}(0,y). \tag{27}$$

Differentiating $R_N(y)$ with respect to y yields

$$\begin{aligned}\frac{\partial R_N(y)}{\partial y} &= \frac{\exp(-\eta^2)}{\lambda^\kappa}\left(-\frac{1}{\lambda}\right)\exp\left(-\frac{y}{\lambda}\right)\sum_{n=N+1}^{\infty} g^{(n)}(0,y)\frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!} \\ &\quad + \frac{\exp(-\eta^2)}{\lambda^\kappa}\exp\left(-\frac{y}{\lambda}\right)\sum_{n=N+1}^{\infty}\frac{\partial g^{(n)}(0,y)}{\partial y}\cdot\frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!} \\ &= \frac{\exp(-\eta^2)}{\lambda^\kappa}\exp\left(-\frac{y}{\lambda}\right)\sum_{n=N+1}^{\infty}\frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!}\left(-\frac{1}{\lambda}g^{(n)}(0,y) + n g^{(n-1)}(0,y)\right). \end{aligned} \tag{28}$$

Furthermore, from (20) we have

$$-\frac{1}{\lambda}g^{(n)}(0,y) + n g^{(n-1)}(0,y) = y^n. \tag{29}$$

By substituting (29) into (28), we have

$$\frac{\partial R_N(y)}{\partial y} = \frac{\exp(-\eta^2)}{\lambda^\kappa}\exp\left(-\frac{y}{\lambda}\right)\sum_{n=N+1}^{\infty}\frac{(\eta^2/\lambda)^{n-\kappa+1}y^n}{n!(n-\kappa+1)!}, \tag{30}$$

which is positive. Since $R_N(y) \le 0$, $\lim_{y\to\infty} R_N(y) = 0$ and $\frac{\partial R_N(y)}{\partial y} > 0$, the residual error magnitude monotonically decreases as y increases and the maximum error occurs at $y = 0$ for any given N.

Now, let us compute the worst truncation error $R_N(0)$, which is given by

$$R_N(0) = \frac{\exp(-\eta^2)}{\lambda^\kappa}\sum_{n=N+1}^{\infty} g^{(n)}(0,0)\frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!}. \tag{31}$$

From (20), we have $g^{(n)}(0,0) = -n!\lambda^{n+1}$. Therefore,

$$\begin{aligned} R_N(0) &= \frac{\exp(-\eta^2)}{\lambda^\kappa}\sum_{n=N+1}^{\infty}(-n!\lambda^{n+1})\frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!} \\ &= -\frac{\exp(-\eta^2)}{\lambda^\kappa}\sum_{n=N+1}^{\infty}\lambda^{n+1}\frac{(\eta^2/\lambda)^{n-\kappa+1}}{(n-\kappa+1)!} \\ &= -\frac{\exp(-\eta^2)}{\lambda^\kappa}\sum_{n=N+1}^{\infty}\frac{(\eta^2)^{n-\kappa+1}}{(n-\kappa+1)!}\cdot\lambda^\kappa \\ &= -\exp(-\eta^2)\sum_{n=N+1}^{\infty}\frac{(\eta^2)^{n-\kappa+1}}{(n-\kappa+1)!}. \end{aligned} \tag{32}$$

From (17), $N \geq \kappa - 2$. For general $N \geq \kappa - 2$, let $m = n - \kappa + 1$. Then,

$$R_N(0) = -\exp(-\eta^2) \sum_{m=N-\kappa+2}^{\infty} \frac{(\eta^2)^m}{m!}.$$

Note that $\sum_{m=N-\kappa+2}^{\infty} \frac{(\eta^2)^m}{m!}$ is the residual error of the Taylor series expansion of $\exp(x)$ after the first $(N - \kappa + 1)$ terms. By the Taylor theorem,

$$\sum_{m=N-\kappa+2}^{\infty} \frac{(\eta^2)^m}{m!} = \frac{(\eta^2)^{N-\kappa+2}}{(N-\kappa+2)!} \exp(\alpha\eta^2) \tag{33}$$

where some $\alpha \in [0, 1]$. Therefore, the worst truncation error is given by

$$|R_N(0)| = \exp\Big((\alpha - 1)\eta^2\Big) \times \frac{(\eta^2)^{N-\kappa+2}}{(N-\kappa+2)!} \leq \frac{(\eta^2)^{N-\kappa+2}}{(N-\kappa+2)!}, \tag{34}$$

where the inequality holds since $\exp((\alpha - 1)\eta^2) \leq 1$ for $0 \leq \alpha \leq 1$. Furthermore, the residual error magnitude is a strictly decreasing function of N for any y,

$$|R_N(y)| > |R_{N+1}(y)|. \tag{35}$$

This can be shown easily as follows.

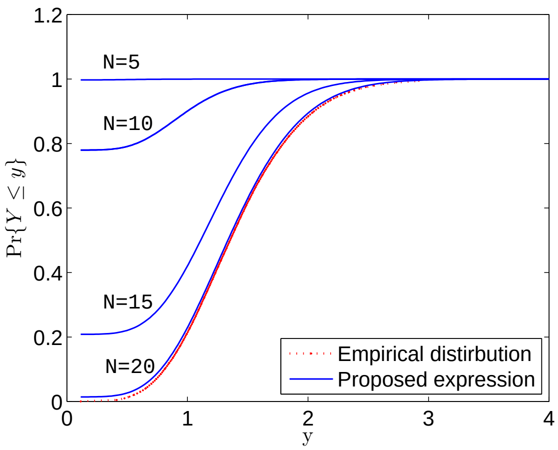
$$\begin{aligned} R_N(y) =& \frac{\exp(-\eta^2)}{\lambda^\kappa} \exp\left(-\frac{y}{\lambda}\right) \sum_{n=N+1}^{\infty} g^{(n)}(0, y) \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!} \\ =& \frac{\exp(-\eta^2)}{\lambda^\kappa} \exp\left(-\frac{y}{\lambda}\right) \left\{ \sum_{n=N+2}^{\infty} g^{(n)}(0, y) \frac{(\eta^2/\lambda)^{n-\kappa+1}}{n!(n-\kappa+1)!} + g^{(N+1)}(0, y) \frac{(\eta^2/\lambda)^{N-\kappa+2}}{(N+1)!(N-\kappa+2)!} \right\} \\ =& R_{N+1}(y) + \frac{\exp(-\eta^2)}{\lambda^\kappa} \exp\left(-\frac{y}{\lambda}\right) \cdot g^{(N+1)}(0, y) \frac{(\eta^2/\lambda)^{N-\kappa+2}}{(N+1)!(N-\kappa+2)!}. \end{aligned}$$

Since $R_N(y) < 0$ and $g^{(N+1)}(y) < 0$ for all $y \geq 0$ and N, we have (35). Now, based on (34) and (35), with given $\boldsymbol{\chi}_k$ and σ_h^2, we can compute the required number N of terms in the series to achieve the desired level of accuracy since η^2 is known.

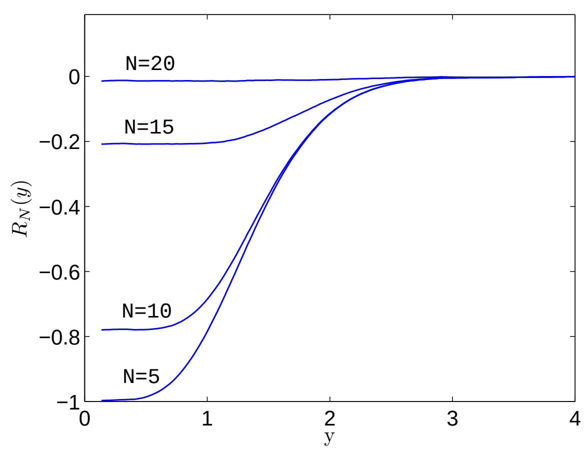
Finally, consider the worst case of $N = \kappa - 2$ and $y = 0$:

$$R_{\kappa-2}(0) = -\exp(-\eta^2) \sum_{n=\kappa-1}^{\infty} \frac{(\eta^2)^{n-\kappa+1}}{(n-\kappa+1)!} = -\exp(-\eta^2) \sum_{m=0}^{\infty} \frac{(\eta^2)^m}{m!} = -1,$$

where the second equality is by replacing $m = n - \kappa + 1$. It is easy to see that the worst case error is -1 in the identity covariance matrix case. Fig. 5 shows the performance of the proposed series expansion in the case of the identity covariance matrix. The numerical results well match our theoretical analysis in this subsection. From the figure, it seems reasonable to choose $N \geq 20 \sim 30$ for accurate outage probability computation.



(a) CDF



(b) Residual error

Fig. 5. number of variables = 4, $\boldsymbol{\mu} = 0.5\mathbf{1}$, $\bar{\mathbf{Q}} = \mathbf{I}$, and $\boldsymbol{\Sigma} = 0.1\mathbf{I}$.